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Global Hopf bifurcation for differential equations with state-dependent delay

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ABSTRACT

We develop a global Hopf bifurcation theory for a system of functional differential equations with state-dependent delay. The theory is based on an application of the homotopy invariance of S^1-equivariant degree using the formal linearization of the system at a stationary state. Our results show that under a set of mild conditions the information about the characteristic equation of the formal linearization with frozen delay can be utilized to detect the local Hopf bifurcation and to describe the global continuation of periodic solutions for such a system with state-dependent delay.

1. Introduction

Our goal is to apply the S^1-equivariant degree theory to describe the occurrence of local Hopf bifurcation from a stationary state, and the global continuation of periodic solutions for the following parameterized functional differential equations (FDEs) with state-dependent delay [1,2,4–6,11,37,38]

$$\begin{pmatrix} \dot{x}(t) \\ \dot{\tau}(t) \end{pmatrix} = \begin{pmatrix} f(x(t), x(t-\tau(t)), \sigma) \\ g(x(t), \tau(t), \sigma) \end{pmatrix}, \tag{1.1}$$

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where $x \in \mathbb{R}^N$, $\tau \in \mathbb{R}$, $t \in \mathbb{R}$ and $\sigma \in \mathbb{R}$, $f : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$, and $g : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given maps. A stationary state of (1.1) with parameter σ is a vector $(x, \tau) \in \mathbb{R}^N \times \mathbb{R}$ so that $f(x, x, \sigma) = 0$ and $g(x, \tau, \sigma) = 0$.

The major problem to develop such a global Hopf bifurcation theory for the system of FDEs (1.1) is that in the spaces of continuous periodic functions $C_T(\mathbb{R}; \mathbb{R}^N) = \{x \in C(\mathbb{R}; \mathbb{R}^N)\colon\ x(t+T) = x(t) \text{ for all } t \in \mathbb{R}\}$ and $C_T(\mathbb{R}; \mathbb{R}) = \{\tau \in C(\mathbb{R}; \mathbb{R})\colon\ \tau(t+T) = \tau(t) \text{ for all } t \in \mathbb{R}\}$ with a fixed period $T > 0$, the composition operator

$$\chi : C_T\big(\mathbb{R}; \mathbb{R}^N\big) \times C_T(\mathbb{R}; \mathbb{R}) \to C_T\big(\mathbb{R}; \mathbb{R}^N\big), \tag{1.2}$$

$$\chi(x, \tau)(t) = x\big(t - \tau(t)\big), \quad t \in \mathbb{R}, \tag{1.3}$$

is generally not a C^1 (continuously differentiable) map with respect to τ in the supremum norm. This causes the difficulty in formulating linearization at a stationary state, and such a linearization is usually necessary in the functional analytical setting for the Hopf bifurcation problem where a topological index such as a S^1-equivariant degree can be calculated and applied to investigate the birth and continuation of periodic solutions bifurcating from a stationary state.

In [12], a system of auxiliary equations obtained through a formal linearization technique was used in the study of local stability of FDEs with state-dependent delays in the space of continuously differentiable functions. This formal linearization technique is only heuristic and can be described in the following way: the state-dependent delay $\tau(t)$ in $x(t - \tau(t))$ is first fixed at a given stationary state, then the resulting nonlinear system with the frozen constant delay is linearized. Other applications of the system of auxiliary equations obtained through a formal linearization process can be found in [8,10,17,19]. None of these results is sufficient for us to develop a global Hopf bifurcation theory based on the S^1-equivariant degree for FDEs (1.1) with state-dependent delay. However, the above mentioned results strongly indicate that the system of auxiliary equations obtained through the heuristic technique of formal linearization can be utilized to detect the local Hopf bifurcation and to describe its global continuation for FDEs with state-dependent delay.

In this paper we use the homotopy invariance property of the S^1-equivariant degree to relate the Hopf bifurcation problem of (1.1) to the change of stability of stationary states of the corresponding system of auxiliary equations obtained through formal linearization. As such we show that information of the auxiliary equations can be used in a standard way to develop a local and global Hopf bifurcation theory for FDEs with state-dependent delay.

We organize the remaining part of this paper as follows. In Section 2, we use the general results in [13,24] of the S^1-equivariant degree to develop a Hopf bifurcation framework for FDEs with state-dependent delay. We then, in Sections 3 and 4, state and prove our main results about the local bifurcation and global continuation of periodic solutions for the parameterized system of FDEs (1.1) with state-dependent delay. Some remarks are given, in the final section, about the novelty and challenge of our approach in comparison with existing studies.

2. S^1-degree and equivariant formulation of Hopf bifurcations

We describe, following [13], the framework where an S^1-equivariant degree can be applied to yield a Hopf bifurcation theory for parameterized dynamical systems. We refer to [24] for relevant concepts of the S^1-equivariant degree.

Let V be a real isometric Banach representation of the group $G = S^1 := \{z \in \mathbb{C};\ |z| = 1\}$. This means that V is a Banach space with an isometric Banach representation of the group G (see [24], p. 194 and p. 196 for details). Then V has the following direct sum decomposition

$$V = \overline{\bigoplus_{k=0}^{+\infty} V_k},$$

where $V_0 = V^G := \{x \in V;\, gx = x \text{ for all } g \in G\}$ is the subspace of G-fixed points, and for $k \geqslant 1$, $x \in V_k \setminus \{0\}$ implies that the isotropy group G_x is $\mathbb{Z}_k := \{\gamma \in G;\, \gamma^k = 1\}$. We call such a decomposition an *isotypical direct sum decomposition*. We assume that

(A1) for each integer $k = 0, 1, \ldots,$ the subspace V_k is of finite dimension.

The subspace V_k, with $k \geqslant 1$, $k \in \mathbb{N}$, is the vector space of all mappings of the form $x \sin k\cdot + y \cos k\cdot$, $x, y \in \mathbb{R}^{N+1}$, and can be endowed with a complex structure $J : V_k \to V_k$ by

$$J(x \cos k\cdot + y \sin k\cdot) = -x \sin k\cdot + y \cos k\cdot. \tag{2.1}$$

Let X_0 be a Banach space and $\mathcal{M}$ be the class of bounded subsets of X_0. The *Kuratowski Measure of Noncompactness* $\alpha : \mathcal{M} \to [0, +\infty)$ is defined for $A \in \mathcal{M}$ by $\alpha(A) = \inf\{\epsilon > 0\colon\ A = \bigcup_{i=1}^n A_i,\ \operatorname{diam}(A_i) < \epsilon \text{ for } i = 1, 2, \ldots, n, \text{ where } n \in \mathbb{N}\}$. A continuous map $F : \mathrm{Dom}(F) \subset X_0 \to X_0$ is called a *condensing map* if $\alpha(F(E)) < \alpha(E)$ for all $E \subseteq \mathrm{Dom}(F)$ and $E \in \mathcal{M}$ with $\alpha(E) > 0$.

Let X, Y be two topological spaces and $\Omega \subseteq X$ be an open, nonempty and bounded set. We say $F : X \to Y$ is *Ω-admissible* if F is continuous and $F(x) \neq 0$ for all $x \in \partial\Omega$ where $\partial\Omega$ is the boundary of Ω. Let I be the closed unit interval $[0, 1]$. Two continuous maps F_1, F_2 from X to Y are called *homotopic* if there exists a continuous map H from $X \times I$ to Y such that $H(x, 0) = F_1(x)$ and $H(x, 1) = F_2(x)$ for all $x \in X$. H is then called a homotopy between F_1 and F_2. If H from $X \times I$ to Y is a homotopy between two Ω-admissible maps F_1 and F_2 from X to Y and if $H(x, t) \neq 0$ for all $(x, t) \in [0, 1] \times \partial\Omega$, then we say F_1 and F_2 are *homotopic* on Ω (or Ω-homotopic) and H is an *Ω-homotopy*.

Let $L_0 : \mathrm{Dom}(L_0) \subseteq V \to V$ be an equivariant linear operator. Here and in what follows, L_0 is *equivariant* means that $L_0(gx) = gL_0(x)$ for every $x \in \mathrm{Dom}(L_0)$ and $g \in G$. A compact linear operator $K : V \to V$ is called a *compact resolvent* of L_0 if $L_0 + K : \mathrm{Dom}(L_0) \to V$ is a bijection. Denote by $CR^G(L_0)$ the set of all equivariant compact resolvents of L_0. We assume $CR^G(L_0) \neq \emptyset$.

We consider the following continuous map $\mathscr{F} : \mathrm{Dom}(L_0) \times \mathbb{R}^2 \subset V \times \mathbb{R}^2 \to V$ given by

$$\mathscr{F}(u, \lambda) = L_0 u - N_0(u, \lambda), \quad (u, \lambda) \in \mathrm{Dom}(L_0) \times \mathbb{R}^2, \tag{2.2}$$

where $N_0 : V \times \mathbb{R}^2 \to V$ is a continuous map that is G-equivariant (i.e., $N_0(gu, \lambda) = gN_0(u, \lambda)$ for $u \in V$, $\lambda \in \mathbb{R}^2$ and $g \in G$) and we have the following assumptions:

(A2) There exists $K \in CR^G(L_0)$ such that for every fixed parameter $\lambda \in \mathbb{R}^2$,

$$(L_0 + K)^{-1} \circ \big[N_0(\cdot, \lambda) + K\big] : V \to V$$

is a condensing map.

(A3) There exists a 2-dimensional submanifold $M \subseteq V_0 \times \mathbb{R}^2$ such that
  (i) $M \subseteq \mathscr{F}^{-1}(0)$;
  (ii) if $(u_0, \lambda_0) \in M$ then there exists an open neighborhood U_{λ_0} of λ in $\mathbb{R}^2$, an open neighborhood U_{u_0} of u_0 in V_0, and a C^1-map $\eta : U_{\lambda_0} \to V_0$ such that

$$M \cap (U_{u_0} \times U_{\lambda_0}) = \big\{\big(\eta(\lambda), \lambda\big);\ \lambda \in U_{\lambda_0}\big\}.$$

In relation to the bifurcation problem of (2.2), we consider the structure of the set of solutions to the following equation

$$\mathscr{F}(u, \lambda) = 0, \quad (u, \lambda) \in \mathrm{Dom}(L_0) \times \mathbb{R}^2. \tag{2.3}$$

All points $(u, \lambda) \in M$ are called *trivial solutions* of (2.2) or (2.3), and all other solutions in $\mathscr{F}^{-1}(0) \setminus M$ are called *nontrivial solutions*. A point $(u_0, \lambda_0) \in M$ is called a bifurcation point if in any neighborhood of $(u_0, \lambda_0) \in M$ there is a nontrivial solution for (2.3).

Eq. (2.3) can be transformed into the equivariant fixed point problem

$$u = R_K \circ \big[K + N_0(\cdot,\lambda)\big](u), \quad (u,\lambda) \in V \times \mathbb{R}^2, \tag{2.4}$$

where $R_K := (L_0 + K)^{-1}$. Let $\mathcal{F}(u,\lambda) = u - R_K \circ [N_0(\cdot,\lambda) + K](u)$, $(u,\lambda) \in V \times \mathbb{R}^2$. Then (2.3) is equivalent to the equation

$$\mathcal{F}(u,\lambda) = 0, \quad (u,\lambda) \in V \times \mathbb{R}^2. \tag{2.5}$$

The idea of finding nontrivial solutions to (2.5) in an open G-invariant neighborhood $\mathcal{U} \subseteq V \times \mathbb{R}^2$ of $(u_0,\lambda_0) \in M$ is based on an *auxiliary function* ψ to (2.5), which is introduced to distinguish nontrivial solutions from trivial solutions. Here, $\mathcal{U}$ is said to be G-invariant if $(gx,\lambda) \in \mathcal{U}$ for all $g \in G$, $(x,\lambda) \in \mathcal{U}$. An auxiliary function to (2.5) on the set $\mathcal{U}$ is an equivariant function (i.e., $\psi(gx) = g\psi(x)$ for all $g \in G$ and $x \in \overline{\mathcal{U}}$, where $\overline{\mathcal{U}}$ denotes the closure of $\mathcal{U}$. Here and in what follows G acts on $\mathbb{R}^2$ trivially) $\psi : \overline{\mathcal{U}} \subset V \times \mathbb{R}^2 \to \mathbb{R}$ satisfying that $\psi(u,\lambda) < 0$ for all $(u,\lambda) \in \overline{\mathcal{U}} \cap M$. Then every solution to the system

$$\begin{cases} \mathcal{F}(u,\lambda) = 0, \\ \psi(u,\lambda) = 0, \end{cases} \quad (u,\lambda) \in \overline{\mathcal{U}} \tag{2.6}$$

is a nontrivial solution to (2.2). This leads to the equivariant map $\mathcal{F}_\psi : \overline{\mathcal{U}} \to V \times \mathbb{R}$ defined by

$$\mathcal{F}_\psi(u,\lambda) = \big(\mathcal{F}(u,\lambda), \psi(u,\lambda)\big), \quad (u,\lambda) \in \overline{\mathcal{U}}, \tag{2.7}$$

and the problem of finding a nontrivial solution to (2.2) in $\mathcal{U}$ can be reduced to the problem of finding a solution to the equation $\mathcal{F}_\psi(u,\lambda) = 0$ in $\mathcal{U}$ which may be solved by using the so-called S^1-degree (see [24] for details) as a topological invariant associated with the problem (2.6). To be more specific, if $\mathcal{F}_\psi(u,\lambda) = 0$ has no solution on $\partial\mathcal{U}$ and $\mathcal{F} : \mathcal{U} \to V$ is a condensing field (i.e. $\pi - \mathcal{F}$ is a condensing map, where $\pi : \mathcal{U} \to V$ is the natural projection on V), then the S^1-equivariant degree $S^1\text{-}\deg(\mathcal{F}_\psi, \mathcal{U})$ is well defined and its nontriviality implies the existence of solutions of $\mathcal{F}_\psi(u,\lambda) = 0$ in $\mathcal{U}$. Global continuation of the branch of nontrivial solutions (solutions in $\mathscr{F}^{-1}(0) \setminus M$) bifurcating from (u_0,λ_0) can be characterized by the above S^1-degree at all bifurcation points along the closure of the branch, if such a branch is bounded in $V \times \mathbb{R}^2$ (the so-called Fuller space).

To describe precisely this S^1-degree based bifurcation theory, we need some additional information about:

(a) the construction of the open neighborhood $\mathcal{U}$;
(b) the auxiliary function ψ;
(c) the computation of $S^1\text{-}\deg(\mathcal{F}_\psi, \mathcal{U})$.

We start with the construction of the open neighborhood $\mathcal{U}$. Usually, if $\mathcal{F}(u,\lambda)$ is differentiable with respect to u, we are able to define *singular points* of system (2.5) through its linearization at the trivial solutions of (2.3). This is unfortunately not so for the Hopf bifurcation problem of (1.1), as explained in the introduction. So, we need to justify that the formal linearization can be utilized to detect the local Hopf bifurcation and to describe the global continuation of periodic solutions for such a system with state-dependent delay. Our approach towards this justification of formal linearization is through a simple homotopy argument. Namely, we will consider, in the context of Hopf bifurcation of (1.1), the following equation

$$\tilde{\mathcal{F}}(u,\lambda) = 0, \quad (u,\lambda) \in \bar{\mathcal{U}} \tag{2.8}$$

for an S^1-equivariant C^1-map $\tilde{\mathcal{F}}:\overline{\mathcal{U}}\to V$ that is S^1-homotopic to $\mathcal{F}$ in a sense to be detailed below. For the functional analytic setting of the Hopf bifurcation of (1.1), such a C^1-map is attained by extending a linear operator obtained through the formal linearization from a C^1-space to a C-space, an idea previously used in [14] and [32] (see the final section for more discussions). To be more precise, we assume that such a C^1-map is given by

$$\tilde{\mathcal{F}}(u,\lambda)=u-R_K\circ\big[\tilde{N}_0(\cdot,\lambda)+K\big](u),\quad (u,\lambda)\in\bar{\mathcal{U}}, \tag{2.9}$$

where $\tilde{N}_0:\overline{\mathcal{U}}\to V$ is an S^1-equivariant C^1-map and

(A4) $M\subseteq\tilde{\mathcal{F}}^{-1}(0)$, and for every $\lambda\in\mathbb{R}^2$, $R_K\circ(\tilde{N}_0(\cdot,\lambda)+K):V\to V$ is a condensing map.

By the Implicit Function Theorem, if $(u_0,\lambda_0)\in M$ is a bifurcation point of system (2.9), then the derivative $D_u\tilde{\mathcal{F}}(u_0,\lambda_0)$ which is G-equivariant is not an automorphism of V. Therefore, all bifurcation points of (2.9) are contained in the set

$$\Lambda:=\big\{(u,\lambda)\in M:\ D_u\tilde{\mathcal{F}}(u,\lambda)\notin GL_G(V)\big\},$$

where $GL_G(V)$ denotes the set of G-equivariant automorphisms of V.

In what follows, we call a point $(u_0,\lambda_0)\in\Lambda$ a *V-singular point* of $\tilde{\mathcal{F}}$. If (u_0,λ_0) is the only V-singular point in some neighborhood of $(u_0,\lambda_0)\in V\times\mathbb{R}^2$, we say that (u_0,λ_0) is an *isolated V-singular point* of $\tilde{\mathcal{F}}$.

Let (u_0,λ_0) be an isolated V-singular point of $\tilde{\mathcal{F}}$. We consider the open neighborhood of $(u_0,\lambda_0)\in M$ defined by

$$B_M(u_0,\lambda_0;r,\rho):=\big\{(u,\lambda)\in V\times\mathbb{R}^2;\ |\lambda-\lambda_0|<\rho,\ \big\|u-\eta(\lambda)\big\|<r\big\}, \tag{2.10}$$

where $\rho>0$ and $r>0$ are chosen so that

(i) $\tilde{\mathcal{F}}(u,\lambda)\neq 0$ for all $(u,\lambda)\in\overline{B_M(u_0,\lambda_0;r,\rho)}$ such that $|\lambda-\lambda_0|=\rho$, $\|u-\eta(\lambda)\|\neq 0$;
(ii) (u_0,λ_0) is the only V-singular point of $\tilde{\mathcal{F}}$ in $\overline{B_M(u_0,\lambda_0;r,\rho)}$.

We call $B_M(u_0,\lambda_0;r,\rho)$ a *special neighborhood* of $\tilde{\mathcal{F}}$ determined by r and ρ. The existence of a special neighborhood $B_M(u_0,\lambda_0;r,\rho)$ follows from the assumption that the V-singular point (u_0,λ_0) of $\tilde{\mathcal{F}}$ is isolated. (e.g., see [24], p. 169). That is, there exists $\rho_0>0$ such that for any $0<\rho<\rho_0$, there exists $r>0$ such that $B_M(u_0,\lambda_0;r,\rho)$ is a special neighborhood of $\tilde{\mathcal{F}}$.

To tie the S^1-equivariant degree of $\mathcal{F}$ to that of $\tilde{\mathcal{F}}$, we assume that

(A5) We can choose the constants $r>0$ and $\rho>0$ so that $B_M(u_0,\lambda_0;r,\rho)$ is a special neighborhood of $\tilde{\mathcal{F}}$, and that there exists $0<r'\leqslant r$ such that $\mathcal{F}(u,\lambda)\neq 0$ for all $(u,\lambda)\in B_M(u_0,\lambda_0;r',\rho)$ with $|\lambda-\lambda_0|=\rho$ and $\|u-\eta(\lambda)\|\neq 0$.

If ψ is an auxiliary function to (2.5), then by the construction of the S^1-degree and the assumptions (A2), (A4) and (A5), there exists a special neighborhood $\mathcal{U}:=B_M(u_0,\lambda_0;r',\rho)$ of $\tilde{\mathcal{F}}$ such that the continuous G-equivariant maps $\mathcal{F}_\psi$ and $\tilde{\mathcal{F}}_\psi$ are nonzero on the boundary of $\mathcal{U}$, and therefore both $S^1\text{-}\deg(\tilde{\mathcal{F}}_\psi,\mathcal{U})$ and $S^1\text{-}\deg(\mathcal{F}_\psi,\mathcal{U})$ are well defined.

Note that the equivariant version of Dugundji's extension theorem (see [24], p. 197) implies that there exists a continuous S^1-equivariant function $\theta:\overline{\mathcal{U}}\to\mathbb{R}$ such that

(i) $\theta(\eta(\lambda),\lambda)=-|\lambda-\lambda_0|$ for all $(\eta(\lambda),\lambda)\in\overline{\mathcal{U}}\cap M$;
(ii) $\theta(u,\lambda)=r'$ if $\|u-\eta(\lambda)\|=r'$.

The function θ is called a *completing function* (or Ize's function). Clearly, if θ is a completing function, then $\psi_\delta(u,\lambda) := \theta(u,\lambda) - \delta$ is negative on the subset of trivial solutions $\mathcal{U} \cap M$, provided that $\delta > 0$. So, ψ_δ is an auxiliary function to (2.5) and (2.8).

For $\delta > 0$ small enough, we can define $\mathcal{F}_{\psi_\delta} : \overline{\mathcal{U}} \to V \times \mathbb{R}$ by

$$\mathcal{F}_{\psi_\delta}(u,\lambda) := \big(\mathcal{F}(u,\lambda), \psi_\delta(u,\lambda)\big),$$

and define the S^1-equivariant degree $S^1\text{-}\deg(\mathcal{F}_{\psi_\delta}, \mathcal{U})$. By the homotopy invariance of the S^1-degree, $S^1\text{-}\deg(\mathcal{F}_{\psi_\delta}, \mathcal{U}) = S^1\text{-}\deg(\mathcal{F}_\theta, \mathcal{U})$. Therefore, the nontriviality of $S^1\text{-}\deg(\mathcal{F}_\theta, \mathcal{U})$ implies the existence of nontrivial solution of (2.3) in $\mathcal{U}$.

We now turn to the computation of $S^1\text{-}\deg(\mathcal{F}_\theta, \mathcal{U})$. If $\tilde{\mathcal{F}}_\theta = (\tilde{\mathcal{F}}, \theta)$ is homotopic to $\mathcal{F}_\theta$ on $\mathcal{U}$, then the homotopy invariance of S^1-degree ensures that $S^1\text{-}\deg(\mathcal{F}_\theta, \mathcal{U}) = S^1\text{-}\deg(\tilde{\mathcal{F}}_\theta, \mathcal{U})$. In the following part of this section, we present an algorithm to calculate $S^1\text{-}\deg(\tilde{\mathcal{F}}_\theta, \mathcal{U})$.

We identify $\mathbb{R}^2$ with $\mathbb{C}$, and for sufficiently small $\rho > 0$, define $\alpha : D \to M$, $D := \{z \in \mathbb{C};\ |z| \leqslant 1\}$, by

$$\alpha(z) = \big(\eta(\lambda_0 + \rho z), \lambda_0 + \rho z\big) \in V_0 \oplus \mathbb{R}^2.$$

Note that $\tilde{\mathcal{F}}$ is a continuous S^1-equivariant map on $\overline{\mathcal{U}}$. The formula $\Psi(z) := D_u\tilde{\mathcal{F}}(\alpha(z))$, $z \in S^1 \subseteq D$, defines a continuous map $\Psi : S^1 \to GL_G(V)$ which has the decomposition (see [13] for details) $\Psi = \Psi_0 \oplus \Psi_1 \oplus \cdots \oplus \Psi_k \oplus \cdots$, where $\Psi_k = \Psi|_{V_k} : S^1 \to GL_G(V_k)$ for $k = 1, 2, \ldots$ and $\Psi_0 : S^1 \to GL(V_0)$ with $GL(V_0)$ being the set of linear automorphisms of V_0. We now define

$$\begin{cases} \epsilon_0(u_0,\lambda_0) = \operatorname{sgn}\det \Psi_0(z), \\ \mu_k(u_0,\lambda_0) = \deg_B\big(\det_{\mathbb{C}}[\Psi_k]\big), \qquad k = 1, 2, \ldots, \\ \mu(u_0,\lambda_0) = \big\{\mu_k(u_0,\lambda_0)\big\} \in \bigoplus_{k=1}^{\infty} \mathbb{Z}, \end{cases} \tag{2.11}$$

where $[\Psi_k]$ is the matrix representation of Ψ_k with respect to an ordered $\mathbb{C}$-basis of V_k and V_k is endowed with the complex structure defined at (2.1), $\det_{\mathbb{C}}(\cdot)$ is the determinant mapping, $\deg_B(\det_{\mathbb{C}}[\Psi_k])$ is the usual Brouwer degree of $\det_{\mathbb{C}}[\Psi_k]$ on $\{z \in \mathbb{C}:\ |z| < 1\}$. It is clear that ϵ_0 does not depend on the choice of $z \in S^1$.

We need one more notion, the crossing numbers, to calculate $\deg_B(\det_{\mathbb{C}}[\Psi_k])$:

Lemma 2.1. *(See [13].) Suppose $\alpha_0, \beta_0, \delta, \varepsilon$ are given numbers with $\alpha_0, \delta, \varepsilon > 0$. Let $\Omega := (0, \alpha_0) \times (\beta_0 - \varepsilon, \beta_0 + \varepsilon) \subseteq \mathbb{R}^2$. Assume $H : [\sigma_0 - \delta, \sigma_0 + \delta] \times \bar{\Omega} \to \mathbb{R}^2$ is a continuous function satisfying*

(i) *$H(\sigma, \alpha, \beta) \neq 0$ for all $\sigma \in [\sigma_0 - \delta, \sigma_0 + \delta]$ and $(\alpha, \beta) \in \partial\Omega \setminus \{(0, \beta);\ \beta \in (\beta_0 - \varepsilon, \beta_0 + \varepsilon)\}$;*
(ii) *if $(\alpha, \beta) \in \Omega$ and $H_{\sigma_0 \pm \delta}(\alpha, \beta) = 0$, then $\alpha \neq 0$.*

Let $\Omega_1 := (\sigma_0 - \delta, \sigma_0 + \delta) \times (\beta_0 - \varepsilon, \beta_0 + \varepsilon)$ and define the function $\Psi_H : \bar{\Omega}_1 \to \mathbb{R}^2$ by $\Psi_H(\sigma, \beta) = H(\sigma, 0, \beta)$, for $\sigma \in [\sigma_0 - \delta, \sigma_0 + \delta]$, and $\beta \in [\beta_0 - \varepsilon, \beta_0 + \varepsilon]$. Then $\Psi_H(\sigma, \beta) \neq 0$ for $(\sigma, \beta) \in \partial\Omega_1$ and $\deg_B(\Psi_H, \Omega_1) = \gamma$, where γ is the crossing number given by

$$\gamma := \deg_B(H_{\sigma_0 - \delta}, \Omega) - \deg_B(H_{\sigma_0 + \delta}, \Omega).$$

Finally, in order to exclude bifurcation of solutions of (2.8) in $V_0 \times \mathbb{R}^2$, we assume that

(A6) $D_u\tilde{\mathcal{F}}(u_0, \lambda_0)|_{V_0} : V_0 \to V_0$ is an isomorphism.

Lemma 2.2. *Assume that* (A1)–(A6) *hold and let* $\mathcal{U} = B_M(u_0, \lambda_0; r', \rho) \subseteq V \times \mathbb{R}^2$ *be a special neighborhood for* $\tilde{\mathcal{F}}$. *If* S^1-$\deg(\tilde{\mathcal{F}}_\theta, \mathcal{U}) \neq 0$ *for some completing function* $\theta : \overline{\mathcal{U}} \to \mathbb{R}$ *and* $\tilde{\mathcal{F}}_\theta$ *is homotopic to* $\mathcal{F}_\theta$ *on* $\overline{\mathcal{U}}$, *then* (u_0, λ_0) *is a bifurcation point for* (2.2). *That is, there exists a sequence of nontrivial solutions* (u_n, λ_n) *of* (2.2) *such that* $\lim_{n\to+\infty}(u_n, \lambda_n) = (u_0, \lambda_0)$.

Proof. Let $\delta > 0$ be sufficiently small so that the function $\psi_\delta : \overline{\mathcal{U}} \to \mathbb{R}$, defined by $\psi_\delta(u, \lambda) = \theta(u, \lambda) - \delta$, $(u, \lambda) \in \overline{\mathcal{U}}$, is an auxiliary function. Let $\theta_t(u, \lambda) = \theta(u, \lambda) - t\delta$, $t \in [0, 1]$. Then we can apply the homotopy invariance of the S^1-degree to obtain

$$S^1\text{-}\deg(\mathcal{F}_{\psi_\delta}, \mathcal{U}) = S^1\text{-}\deg(\mathcal{F}_\theta, \mathcal{U}).$$

By assumption that $\tilde{\mathcal{F}}_\theta$ is homotopic to $\mathcal{F}_\theta$ on $\overline{\mathcal{U}}$, we have

$$S^1\text{-}\deg(\mathcal{F}_\theta, \mathcal{U}) = S^1\text{-}\deg(\tilde{\mathcal{F}}_\theta, \mathcal{U}).$$

Therefore,

$$S^1\text{-}\deg(\mathcal{F}_{\psi_\delta}, \mathcal{U}) = S^1\text{-}\deg(\tilde{\mathcal{F}}_\theta, \mathcal{U}).$$

Thus, S^1-$\deg(\tilde{\mathcal{F}}_\theta, \mathcal{U}) \neq 0$ implies that the equation $\mathcal{F}_{\psi_\delta}(u, \lambda) = 0$ has a solution in $\mathcal{U}$ and hence (2.2) has a nontrivial solution in $\mathcal{U}$. By the excision property of the S^1-degree, we know that S^1-$\deg(\tilde{\mathcal{F}}_\theta, \mathcal{U})$ is independent of the choice of r' and ρ, (e.g., see Proposition 5.1.6 in [24] for details). Therefore, the result follows. □

Lemma 2.3. *Assume that* (A4)–(A6) *hold. Let* $\mathcal{U} = B_M(u_0, \lambda_0; r', \rho) \subseteq V \times \mathbb{R}^2$ *be a special neighborhood of* $\tilde{\mathcal{F}}$, *and* θ *a completing function. Then the* S^1 *degree* $S^1\text{-}\deg(\tilde{\mathcal{F}}_\theta, \mathcal{U})$ *is well defined and*

$$S^1\text{-}\deg(\tilde{\mathcal{F}}_\theta, \mathcal{U}) = \epsilon_0 \cdot \mu(u_0, \lambda_0).$$

That is,

$$S^1\text{-}\deg_k(\tilde{\mathcal{F}}_\theta, \mathcal{U}) = \epsilon_0 \cdot \mu_k(u_0, \lambda_0), \quad k = 1, 2, \ldots,$$

where $\mu(u_0, \lambda_0)$ *is defined by* (2.11).

Proof. Note that by the assumptions (A4) and (A5) and from the construction of the S^1-equivariant degree, $S^1\text{-}\deg(\tilde{\mathcal{F}}_\theta, \mathcal{U})$ is well defined. Then the calculation formula of $S^1\text{-}\deg(\tilde{\mathcal{F}}_\theta, \mathcal{U})$ is a straightforward consequence of Theorem 7.1.5 in [24]. This completes the proof. □

By Lemma 2.2 and Lemma 2.3, we have the following local bifurcation theorem of Krasnosel'skii type.

Theorem 2.4. *Assume that* (A1)–(A6). *Let* $\mathcal{U} = B_M(u_0, \lambda_0; r', \rho) \subseteq V \times \mathbb{R}^2$ *be a special neighborhood of* $\tilde{\mathcal{F}}$, *and* θ *a completing function. If* $\tilde{\mathcal{F}}_\theta$ *is homotopic to* $\mathcal{F}_\theta$ *on* $\overline{\mathcal{U}}$ *and if there exists* $k \geqslant 1$ *such that* $\mu_k(u_0, \lambda_0) \neq 0$, *then* (u_0, λ_0) *is a bifurcation point of* (2.2). *More precisely, there exists a sequence* (u_n, λ_n) *of nontrivial solutions to* (2.2) *such that the isotropy group of* u_n *contains* $\mathbb{Z}_k$ *and* $(u_n, \lambda_n) \to (u_0, \lambda_0)$ *as* $n \to \infty$.

For global bifurcation, we assume further that both $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are defined on $V \times \mathbb{R}^2$, and that

(A7) every bifurcation point of (2.2) is a V-singular point of $\tilde{\mathcal{F}}$.

(A8) $\tilde{\mathcal{F}}_\theta$ is homotopic to $\mathcal{F}_\theta$ on some special neighborhood $\mathcal{U}$ of each isolated V-singular point of $\tilde{\mathcal{F}}$, where θ is a completing function defined on $\mathcal{U}$.

Now we can state and prove the following global bifurcation theorem of Rabinowitz type.

Theorem 2.5. *Assume* (A1)–(A8) *hold and* (A5)–(A6) *hold for every V-singular point (u_0, λ_0) of $\tilde{\mathcal{F}}$. Assume further that every V-singular point of $\tilde{\mathcal{F}}$ in M is isolated and M is complete. Let $\mathcal{S}$ denote the closure of the set of all nontrivial solutions of* (2.2)*. Then for each bounded component C of $\mathcal{S}$, the set $C \cap M$ is a finite set, i.e., $C \cap M = \{(u_1, \lambda_1), (u_2, \lambda_2), \dots, (u_q, \lambda_q)\}$, and*

$$\sum_{i=1}^{q} S^1\text{-}\deg(\mathcal{F}_{\theta_i}, \mathcal{U}_i) = \sum_{i=1}^{q} \epsilon_i \cdot \mu(u_i, \lambda_i) = 0,$$

where $\mathcal{U}_i$ is a special neighborhood of (u_i, λ_i), θ_i is a completing function defined on $\overline{\mathcal{U}}_i$, ϵ_i and $\mu(u_i, \lambda_i)$ are defined by (2.11)*.*

Proof. C is a bounded component of $\mathcal{S}$, then every point of $C \cap M$ is a bifurcation point which is also a V-singular point of $\tilde{\mathcal{F}}$. Since every V-singular point of $\tilde{\mathcal{F}}$ is isolated and M is complete, $C \cap M$ is a bounded and closed subset of $V_0 \times \mathbb{R}^2 \supset M$. By (A1), $V_0 \times \mathbb{R}^2$ is finite dimensional and hence $C \cap M$ is compact. Therefore, $C \cap M$ is a finite set.

Choose $r, \rho > 0$ sufficiently small so that for each $i = 1, 2, \dots, q$, $U_i = B_M(u_i, \lambda_i; r, \rho)$ is a special neighborhood of (u_i, λ_i) for $\tilde{\mathcal{F}}$ and $U_i \cap U_j = \emptyset$ if $i \neq j$. By assumption (A5), we can assume r is also small enough such that $\mathcal{F}(u, \lambda) \neq 0$ for all $(u, \lambda) \in \{(u, \lambda) \in B_M(u_i, \lambda_i; r, \rho); |\lambda - \lambda_0| = \rho, \|u - \eta(\lambda)\| \neq 0\}$.

Let $U = U_1 \cup U_2 \cup \dots \cup U_q$ and find a bounded, open set $\Omega_1 \subset V \times \mathbb{R}^2$ such that $C \setminus U \subseteq \Omega_1$ and $\Omega_1 \cap M = \emptyset$. Put $\Omega_2 = U \cup \Omega_1$, then $C \subseteq \Omega_2$. We can (e.g., see [24], p. 174) find an open, invariant subset $\Omega \subseteq V \times \mathbb{R}^2$ such that $C \subseteq \Omega \subseteq \Omega_2$ and $\partial\Omega \cap \mathcal{S} = \emptyset$.

Note that Ω is an open, bounded, invariant subset. We now choose $r_0 \in (0, r)$ and $\rho_0 \in (0, \rho)$ such that, for every $i = 1, 2, \dots, q$, we have

(i) $B_M(u_i, \lambda_i; r_0, \rho_0) \subseteq \Omega$;
(ii) $\mathcal{U}_i := B_M(u_i, \lambda_i; r_0, \rho_0)$ is a special neighborhood of (u_i, λ_i) for $\tilde{\mathcal{F}}$.

Set $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \dots \cup \mathcal{U}_q$ and

$$\partial\mathcal{U}_{r_0} := \bigl\{(u, \lambda) \in \bar{\Omega}\colon \bigl\|u - \eta(\lambda)\bigr\| = r_0,\ \bigl(\eta(\lambda), \lambda\bigr) \in \overline{\mathcal{U}} \cap M\bigr\}.$$

We note that $r_0 > 0$ and define an invariant function by

$$\theta(u, \lambda) = \begin{cases} |\lambda - \lambda_i|\frac{\|u - \eta(\lambda)\| - r_0}{r_0} + \|u - \eta(\lambda)\|, & \text{if } (u, \lambda) \in \bar{\mathcal{U}}_i, \\ r_0, & \text{if } (u, \lambda) \in C \setminus \mathcal{U}. \end{cases} \tag{2.12}$$

$\mathcal{U}_i$ is a special neighborhood and hence we have $(C \setminus \mathcal{U}) \cap \bar{\mathcal{U}}_i = C \cap \partial\mathcal{U}_i \subseteq \partial\mathcal{U}_{r_0}$, where we have $\theta(u, \lambda) = r_0$. Then, by (2.12), $\theta(u, \lambda)$ is continuous on $(C \setminus \mathcal{U}) \cap \bar{\mathcal{U}}_i$. Also, by the construction of $\mathcal{U}_i$, we have $\bar{\mathcal{U}}_i \cap \bar{\mathcal{U}}_j = \emptyset$ if $i \neq j$. Therefore, $\theta : C \cup \bar{\mathcal{U}} \to \mathbb{R}$ is continuous.

By the equivariant version of Dugundji's extension theorem (see [24], pp. 197), we can extend $\theta : C \cup \bar{\mathcal{U}} \to \mathbb{R}$ to a continuous invariant function $\theta : \overline{\Omega} \to \mathbb{R}$ such that

(iii) $\theta(u, \lambda) = -|\lambda - \lambda_i|$ if $(u, \lambda) \in \overline{\mathcal{U}}_i \cap M$;
(iv) $\theta(u, \lambda) = r_0$ if $(u, \lambda) \in (C \setminus \mathcal{U}) \cup \partial\mathcal{U}_{r_0}$.

Let $\mathcal{F}_\theta(u,\lambda) = (\mathcal{F}(u,\lambda), \theta(u,\lambda))$ and $\tilde{\mathcal{F}}_\theta(u,\lambda) = (\tilde{\mathcal{F}}(u,\lambda), \theta(u,\lambda))$. Then $\mathcal{F}_\theta^{-1}(0) = \mathcal{F}^{-1}(0) \cap \theta^{-1}(0)$. By (iii), we know $\mathcal{F}_\theta^{-1}(0) \subseteq C$. Since $C \cap \partial\Omega = \emptyset$, $\mathcal{F}_\theta^{-1}(0) \cap \partial\Omega = \emptyset$. Therefore, $S^1\text{-}\deg(\mathcal{F}_\theta, \Omega)$ is well defined.

We now construct a homotopy $H : \overline{\Omega} \times [0,1] \to V \times \mathbb{R}$ as follows

$$H(u,\lambda,\alpha) = \big(\mathcal{F}(u,\lambda), (1-\alpha)\theta(u,\lambda) - \alpha\rho_0\big), \quad (u,\lambda,\alpha) \in \overline{\Omega} \times [0,1].$$

Note that trivial solutions $(u,\lambda) \in \bar{\Omega}$ outside S are contained in $\bar{\mathcal{U}}_i \cap M$ for some $i = 1,2,\ldots,q$, and by (iii), we have

$$(1-\alpha)\theta(u,\lambda) - \alpha\rho_0 = -(1-\alpha)|\lambda - \lambda_i| - \alpha\rho_0 < 0.$$

Then, by the fact that $\partial\Omega \cap \mathcal{S} = \emptyset$, we have $H(u,\lambda,\alpha) \neq 0$ for all $(u,\lambda,\alpha) \in \partial\Omega \times [0,1]$. Note that θ is invariant and $\mathcal{F}$ is equivariant. So H is an S^1-homotopy. Since $H(u,\lambda,0) = \mathcal{F}_\theta(u,\lambda)$ and $H(u,\lambda,1) = (\mathcal{F}(u,\lambda), -\rho_0) \neq 0$ for all $(u,\lambda) \in \overline{\Omega} \times [0,1]$, by the existence and homotopy invariance of the S^1-degree, we have $S^1\text{-}\deg(\mathcal{F}_\theta, \Omega) = 0$. But (i)–(iv) imply that $\mathcal{F}_\theta^{-1}(0) \subseteq C \cap \mathcal{U}$. Then it follows from the excision property of the S^1-degree that

$$S^1\text{-}\deg(\mathcal{F}_\theta, \mathcal{U}) = S^1\text{-}\deg(\mathcal{F}_\theta, \Omega) = 0.$$

On the other hand, by the additivity property of the S^1-degree, we have

$$\sum_{i=1}^{q} S^1\text{-}\deg(\mathcal{F}_\theta, \mathcal{U}_i) = S^1\text{-}\deg(\mathcal{F}_\theta, \mathcal{U}) = 0.$$

Let $\theta_i(u,\lambda) = \theta(u,\lambda)|_{\overline{\mathcal{U}}_i}$. Note that $\mathcal{U} \subseteq \overline{\Omega}$ implies that $((C \setminus \mathcal{U}) \cup \partial\mathcal{U}_{r_0}) \cap \overline{\mathcal{U}}_i = \partial\mathcal{U}_i \cap \partial\mathcal{U}_{r_0}$, then $\theta_i(u,\lambda)$ is a completing function on $\overline{\mathcal{U}}_i$ and we have

$$\sum_{i=1}^{q} S^1\text{-}\deg(\mathcal{F}_{\theta_i}, \mathcal{U}_i) = S^1\text{-}\deg(\mathcal{F}_\theta, \mathcal{U}) = 0.$$

By assumption (A8), we have $S^1\text{-}\deg(\mathcal{F}_{\theta_i}, \mathcal{U}_i) = S^1\text{-}\deg(\tilde{\mathcal{F}}_{\theta_i}, \mathcal{U}_i)$. Therefore, it follows from Lemma 2.3 that

$$\sum_{i=1}^{q} \epsilon_i \cdot \mu(u_i,\lambda_i) = \sum_{i=1}^{q} S^1\text{-}\deg(\mathcal{F}_{\theta_i}, \mathcal{U}_i) = 0. \qquad \square$$

3. Local Hopf bifurcation for FDEs with state-dependent delay

We turn to the Hopf bifurcation of (1.1), with its solution denoted by $u(t) = (x(t), \tau(t))$. Denote by $C(\mathbb{R};\mathbb{R}^N)$ the normed space of continuous functions from $\mathbb{R}$ to $\mathbb{R}^N$ equipped with the usual supremum norm $\|x\| = \sup_{t\in\mathbb{R}} |x(t)|$ for $x \in C(\mathbb{R};\mathbb{R}^N)$, where $|\cdot|$ denotes the Euclidean norm. We also denote by $C^1(\mathbb{R};\mathbb{R}^N)$ the normed space of continuously differentiable bounded functions from $\mathbb{R}$ to $\mathbb{R}^N$ equipped with the usual C^1 norm

$$\|x\|_{C^1} = \max\Big\{\sup_{t\in\mathbb{R}}\big|x(t)\big|, \sup_{t\in\mathbb{R}}\big|\dot{x}(t)\big|\Big\}$$

for $x \in C(\mathbb{R};\mathbb{R}^N)$. For a stationary state (u_0, τ_0) of (1.1) with the parameter σ_0, we say (u_0, σ_0) is a Hopf bifurcation point of system (1.1), if there exist a sequence $\{(u_k, \sigma_k, T_k)\}_{k=1}^{+\infty} \subseteq C(\mathbb{R};\mathbb{R}^{N+1}) \times \mathbb{R}^2$ and $T_0 > 0$ such that

$$\lim_{k\to+\infty} \left\|(u_k, \sigma_k, T_k) - (u_0, \sigma_0, T_0)\right\|_{C(\mathbb{R};\mathbb{R}^{N+1})\times\mathbb{R}^2} = 0,$$

and (u_k, σ_k) is a nonconstant T_k-periodic solution of system (1.1).

We assume that

(S1) The map $f : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \ni (\theta_1, \theta_2, \sigma) \to f(\theta_1, \theta_2, \sigma) \in \mathbb{R}^N$ and the map $g : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \ni (\gamma_1, \gamma_2, \sigma) \to g(\gamma_1, \gamma_2, \sigma) \in \mathbb{R}$ are C^2 (twice continuously differentiable).

(S2) There exists $L > 0$ such that $g(\gamma_1, \gamma_2, \sigma) < \frac{L}{L+1}$ for $\gamma_1 \in \mathbb{R}^N$, $\gamma_2 \in \mathbb{R}$, $\sigma \in \mathbb{R}$.

In what follows, we write $\partial_i f = \frac{\partial}{\partial\theta_i} f$ for $i = 1, 2$, and similarly we define $\partial_i g$ for $i = 1, 2$.

As outlined in Section 2, we shall study the Hopf bifurcation of (1.1) through its formal linearization. We assume that for a fixed $\sigma_0 \in \mathbb{R}$, $(x_{\sigma_0}, \tau_{\sigma_0})$ (or, abusing notations, $(x_{\sigma_0}, \tau_{\sigma_0}, \sigma_0)$) is a stationary state of (1.1). That is,

$$f(x_{\sigma_0}, x_{\sigma_0}, \sigma_0) = 0, \qquad g(x_{\sigma_0}, \tau_{\sigma_0}, \sigma_0) = 0.$$

We also assume that

(S3) $(\frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial\theta_2}) f(\theta_1, \theta_2, \sigma)|_{\sigma=\sigma_0,\theta_1=\theta_2=x_{\sigma_0}}$ is nonsingular and

$$\frac{\partial}{\partial\gamma_2} g(\gamma_1, \gamma_2, \sigma)\Big|_{\sigma=\sigma_0,\gamma_1=x_{\sigma_0},\gamma_2=\tau_{\sigma_0}} \neq 0.$$

This assumption implies that there exists $\epsilon_0 > 0$ and a C^1-smooth curve $(\sigma_0 - \epsilon_0, \sigma_0 + \epsilon_0) \ni \sigma \mapsto (x_\sigma, \tau_\sigma) \in \mathbb{R}^{N+1}$ such that (x_σ, τ_σ) is the unique stationary state of (1.1) in a small neighborhood of $(x_{\sigma_0}, \tau_{\sigma_0})$ for σ close to σ_0.

We now consider, for $\sigma \in (\sigma_0 - \epsilon_0, \sigma_0 + \epsilon_0)$, the following formal linearization of system (1.1) at the stationary point $\eta(\sigma) = (x_\sigma, z_\sigma)$:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{\tau}(t) \end{pmatrix} = \begin{bmatrix} \partial_1 f(\sigma) & 0 \\ \partial_1 g(\sigma) & \partial_2 g(\sigma) \end{bmatrix} \begin{pmatrix} x(t) - x_\sigma \\ \tau(t) - \tau_\sigma \end{pmatrix} + \begin{bmatrix} \partial_2 f(\sigma) & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x(t - \tau_\sigma) - x_\sigma \\ \tau(t - \tau_\sigma) - \tau_\sigma \end{pmatrix}, \tag{3.1}$$

where

$$\partial_1 f(\sigma) := \partial_1 f(x_\sigma, \tau_\sigma, \sigma), \qquad \partial_2 f(\sigma) := \partial_2 f(x_\sigma, \tau_\sigma, \sigma),$$
$$\partial_1 g(\sigma) := \partial_1 g(x_\sigma, \tau_\sigma, \sigma), \qquad \partial_2 g(\sigma) := \partial_2 g(x_\sigma, \tau_\sigma, \sigma).$$

Let the state and delay pair

$$u(t) := \big(x(t), \tau(t)\big) = e^{\omega t} \cdot C + (x_\sigma, \tau_\sigma),$$

with $C \in \mathbb{R}^{N+1}$. Then we obtain the following characteristic equation of the linear system corresponding to the inhomogeneous linear system (3.1),

$$\det \Delta_{(x_\sigma,\tau_\sigma,\sigma)}(\omega) = 0, \tag{3.2}$$

where $\Delta_{(x_\sigma,\tau_\sigma,\sigma)}(\omega)$ is an $(N+1)\times(N+1)$ complex matrix defined by

$$\Delta_{(x_\sigma,\tau_\sigma,\sigma)}(\omega)=\omega I-\begin{bmatrix}\partial_1 f(\sigma) & 0\\ \partial_1 g(\sigma) & \partial_2 g(\sigma)\end{bmatrix}-\begin{bmatrix}\partial_2 f(\sigma) & 0\\ 0 & 0\end{bmatrix}e^{-\omega\tau_\sigma}. \tag{3.3}$$

A solution ω_0 to the characteristic equation (3.2) is called a characteristic value of the stationary state $(x_{\sigma_0},\tau_{\sigma_0},\sigma_0)$. $(x_{\sigma_0},\tau_{\sigma_0},\sigma_0)$ is a nonsingular stationary state if and only if zero is not a characteristic value of $(x_{\sigma_0},\tau_{\sigma_0},\sigma_0)$. We say that $(x_{\sigma_0},\tau_{\sigma_0},\sigma_0)$ is a *center* if the set of nonzero purely imaginary characteristic values of $(x_{\sigma_0},\tau_{\sigma_0},\sigma_0)$ is nonempty and discrete. $(x_{\sigma_0},\tau_{\sigma_0},\sigma_0)$ is called an *isolated center* if it is the only center in some neighborhood of $(x_{\sigma_0},\tau_{\sigma_0},\sigma_0)$ in $\mathbb{R}^{N+1}\times\mathbb{R}$.

If $(x_{\sigma_0},\tau_{\sigma_0},\sigma_0)$ is an isolated center of (3.1), then there exist $\beta_0>0$ and $\delta\in(0,\epsilon_0)$ such that

$$\det\Delta_{(x_{\sigma_0},\tau_{\sigma_0},\sigma_0)}(i\beta_0)=0,$$

and

$$\det\Delta_{(x_\sigma,\tau_\sigma,\sigma)}(i\beta)\neq 0, \tag{3.4}$$

for any $\sigma\in(\sigma_0-\delta,\sigma_0+\delta)$ and any $\beta\in(0,+\infty)\setminus\{\beta_0\}$. Hence, we can choose constants $\alpha_0=\alpha_0(\sigma_0,\beta_0)>0$ and $\varepsilon=\varepsilon(\sigma_0,\beta_0)>0$ such that the closure of the set $\Omega:=(0,\alpha_0)\times(\beta_0-\varepsilon,\beta_0+\varepsilon)\subset\mathbb{R}^2\cong\mathbb{C}$ contains no other zero of $\det\Delta_{(x_{\sigma_0},\tau_{\sigma_0},\sigma_0)}(\cdot)$ in $\partial\Omega$. We note that $\det\Delta_{(x_\sigma,\tau_\sigma,\sigma)}(\omega)$ is analytic in ω and is continuous in σ. If $\delta>0$ is small enough, then there is no zero of $\det\Delta_{(x_{\sigma_0\pm\delta},\tau_{\sigma_0\pm\delta},\sigma_0\pm\delta)}(\omega)$ in $\partial\Omega$. So we can define the number

$$\gamma_\pm(x_{\sigma_0},\tau_{\sigma_0},\sigma_0,\beta_0)=\deg_B\bigl(\det\Delta_{(x_{\sigma_0\pm\delta},\tau_{\sigma_0\pm\delta},\sigma_0\pm\delta)}(\cdot),\Omega\bigr),$$

and the crossing number of $(x_{\sigma_0},\tau_{\sigma_0},\sigma_0,\beta_0)$ as

$$\gamma(x_{\sigma_0},\tau_{\sigma_0},\sigma_0,\beta_0)=\gamma_--\gamma_+. \tag{3.5}$$

To formulate the Hopf bifurcation problem as a fixed point problem in the space of continuous functions of period 2π, we normalize the period of the $2\pi/\beta$-periodic solution (x,τ) in (1.1) by $(x(t),\tau(t))=(y(\beta t),z(\beta t))$ and obtain

$$\begin{pmatrix}\dot y(t)\\ \dot z(t)\end{pmatrix}=\frac{1}{\beta}N_0(u,\sigma,\beta)(t), \tag{3.6}$$

where $u=(y,z)$ and $N_0: V\ni(u,\sigma,\beta)\times\mathbb{R}^2\to N_0(u,\sigma,\beta)\in V$, with $V:=C_{2\pi}(\mathbb{R};\mathbb{R}^{N+1})$ equipped with the supremum norm, is defined by

$$N_0(u,\sigma,\beta)(t)=\begin{pmatrix}f(y(t),y(t-\beta z(t)),\sigma)\\ g(y(t),z(t),\sigma)\end{pmatrix}.$$

Correspondingly, (3.1) is transformed into

$$\begin{pmatrix}\dot y(t)\\ \dot z(t)\end{pmatrix}=\frac{1}{\beta}\tilde N_0(u,\sigma,\beta)(t), \tag{3.7}$$

where $(y_\sigma, z_\sigma) = (x_\sigma, \tau_\sigma)$ and $\tilde{N}_0 : V \ni (u, \sigma, \beta) \times \mathbb{R}^2 \to \tilde{N}_0(u, \sigma, \beta) \in V$ is defined by

$$\tilde{N}_0(u, \sigma, \beta)(t) = \begin{bmatrix} \partial_1 f(\sigma) & 0 \\ \partial_1 g(\sigma) & \partial_2 g(\sigma) \end{bmatrix} \begin{pmatrix} y(t) - y_\sigma \\ z(t) - z_\sigma \end{pmatrix} + \begin{bmatrix} \partial_2 f(\sigma) & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} y(t - \beta z_\sigma) - y_\sigma \\ z(t - \beta z_\sigma) - z_\sigma \end{pmatrix}.$$

It is clear that $u := (y, z)$ is 2π-periodic if and only if (x, τ) is $(2\pi/\beta)$-periodic.

Before we state and prove our local Hopf bifurcation theorem, we need some technical preparations. We denote by $C^1_{2\pi}(\mathbb{R}; \mathbb{R}^{N+1})$ the Banach space of 2π-periodic and continuously differentiable functions equipped with the C^1 norm

$$\|x\|_{C^1} = \max\Big\{ \sup_{t\in[0,2\pi]} \big|x(t)\big|, \sup_{t\in[0,2\pi]} \big|\dot{x}(t)\big| \Big\}.$$

Lemma 3.1. *Let $L_0 : C^1_{2\pi}(\mathbb{R}; \mathbb{R}^{N+1}) \to V$ be defined by $L_0 u(t) = \dot{u}(t)$, $t \in \mathbb{R}$ and let $K : V \to \mathbb{R}^{N+1}$ be defined by $Ku(t) = \frac{1}{2\pi}\int_0^{2\pi} u(t)\,dt$, $t \in \mathbb{R}$. Then $L_0 + K$ has a compact inverse $(L_0 + K)^{-1} : V \to V$.*

Proof. We first show the existence of $(L_0 + K)^{-1} : V \to C^1_{2\pi}(\mathbb{R}; \mathbb{R}^{N+1})$. We show that the linear operator $L_0 + K : C^1_{2\pi}(\mathbb{R}; \mathbb{R}^{N+1}) \to V$ is one-to-one and onto. Suppose $(L_0 + K)u = 0$, then $\dot{u}(t) + \frac{1}{2\pi}\int_0^{2\pi} u(s)\,ds = 0$ for $t \in \mathbb{R}$. Therefore,

$$u(t) = -\frac{t}{2\pi}\int_0^{2\pi} u(s)\,ds + u(0). \tag{3.8}$$

Noting that $u(2\pi) = u(0)$, it follows from (3.8) that $\int_0^{2\pi} u(s)\,ds = 0$ and $u(t) \equiv u(0)$. Hence, $u(t) \equiv 0$. This shows that $L_0 + K : C^1_{2\pi}(\mathbb{R}; \mathbb{R}^{N+1}) \to V$ is one-to-one.

Now we show that the operator $L_0 + K : C^1_{2\pi}(\mathbb{R}; \mathbb{R}^{N+1}) \to V$ is onto. For any $v \in V$, integrating both sides of the equation $\dot{u}(t) + \frac{1}{2\pi}\int_0^{2\pi} u(t)\,dt = v(t)$ from 0 to t gives

$$u(t) = u(0) + \int_0^t v(s)\,ds - \frac{t}{2\pi}\int_0^{2\pi} u(s)\,ds. \tag{3.9}$$

Letting $t = 2\pi$ in (3.9) and note that $u(2\pi) = u(0)$, we have

$$\int_0^{2\pi} v(s)\,ds = \int_0^{2\pi} u(s)\,ds. \tag{3.10}$$

Then by (3.9) and (3.10), we have

$$u(t) = u(0) + \int_0^t v(s)\,ds - \frac{t}{2\pi}\int_0^{2\pi} v(s)\,ds. \tag{3.11}$$

Integrating both sides of (3.11) from 0 to 2π, we have

$$\begin{aligned}\int_0^{2\pi} u(t)\,dt &= u(0)\cdot 2\pi + \int_0^{2\pi}\int_0^{t} v(s)\,ds\,dt - \frac{2\pi}{2}\int_0^{2\pi} v(s)\,ds \\ &= u(0)\cdot 2\pi + \int_0^{2\pi}\int_s^{2\pi} v(s)\,dt\,ds - \frac{2\pi}{2}\int_0^{2\pi} v(s)\,ds \\ &= u(0)\cdot 2\pi + \frac{2\pi}{2}\int_0^{2\pi} v(s)\,ds - \int_0^{2\pi} s v(s)\,ds. \end{aligned} \tag{3.12}$$

Then, by (3.10) and (3.12) we have

$$u(0) = \frac{1-\pi}{2\pi}\int_0^{2\pi} v(s)\,ds + \frac{1}{2\pi}\int_0^{2\pi} s v(s)\,ds. \tag{3.13}$$

By (3.11) and (3.13) we have

$$u(t) = \int_0^{t} v(s)\,ds + \frac{1-\pi-t}{2\pi}\int_0^{2\pi} v(s)\,ds + \frac{1}{2\pi}\int_0^{2\pi} s v(s)\,ds. \tag{3.14}$$

That is, for every $v \in V$, we define $u : \mathbb{R} \to \mathbb{R}^{N+1}$ by (3.14). Then u is 2π-periodic since v is 2π-periodic. It is clear from (3.14) that $u \in C^1_{2\pi}(\mathbb{R};\mathbb{R}^{N+1})$ holds. This shows that u is a preimage of v and the operator $L_0 + K : C^1_{2\pi}(\mathbb{R};\mathbb{R}^{N+1}) \to V$ is onto. Then $L_0 + K : C^1_{2\pi}(\mathbb{R};\mathbb{R}^{N+1}) \to V$ is one-to-one and onto. Hence the linear operator $(L_0 + K)^{-1} : V \to C^1_{2\pi}(\mathbb{R};\mathbb{R}^{N+1})$ exists and is given by

$$(L_0 + K)^{-1}(v)(t) = \int_0^{t} v(s)\,ds + \frac{1-\pi-t}{2\pi}\int_0^{2\pi} v(s)\,ds + \frac{1}{2\pi}\int_0^{2\pi} s v(s)\,ds.$$

Next, we show that $(L_0 + K)^{-1} : V \to C^1_{2\pi}(\mathbb{R};\mathbb{R}^{N+1})$ is continuous. Indeed, we have

$$\|L_0 + K\|_{L(C^1_{2\pi}(\mathbb{R};\mathbb{R}^{N+1}),V)} = \sup_{\|u\|_{C^1}=1}\ \sup_{t\in[0,2\pi]}\left|\dot u(t) + \frac{1}{2\pi}\int_0^{2\pi} u(t)\,dt\right| \leqslant 2,$$

which implies that $L_0 + K : C^1_{2\pi}(\mathbb{R};\mathbb{R}^{N+1}) \to V$ is continuous. Then by the Open Mapping Theorem, $(L_0 + K)^{-1} : V \to C^1_{2\pi}(\mathbb{R};\mathbb{R}^{N+1})$ is continuous.

We also note that the embedding $C^1_{2\pi}(\mathbb{R};\mathbb{R}^{N+1}) \hookrightarrow V$ is compact, and hence $(L_0 + K)^{-1} : V \to V$ is a compact linear operator. $\square$

Lemma 3.2. *For any $\sigma \in \mathbb{R}$ and $\beta > 0$, the map $N_0(\cdot,\sigma,\beta) : V \to V$ defined by* (3.6) *is continuous.*

Proof. Let $\{(y_n, z_n)\}_{n=1}^{\infty} \subset V$ be such that $(y_n, z_n) \to (y_0, z_0)$ in V as $n \to \infty$. Then by the Integral Mean Value Theorem (see [25], p. 341), we have

$$
\begin{aligned}
&\big\| N_0(y_n, z_n, \sigma, \beta) - N_0(y_0, z_0, \sigma, \beta) \big\| \\
&\quad \leqslant \sup_{t \in [0, 2\pi]} \big| f\big(y_n(t), y_n\big(t - \beta z_n(t)\big), \sigma\big) - f\big(y_0(t), y_0\big(t - \beta z_0(t)\big), \sigma\big) \big| \\
&\qquad + \sup_{t \in [0, 2\pi]} \big| g\big(y_n(t), z_n(t), \sigma\big) - g\big(y_0(t), z_0(t), \sigma\big) \big| \\
&\quad \leqslant \sup_{t \in [0, 2\pi]} \big| f\big(y_n(t), y_n\big(t - \beta z_n(t)\big), \sigma\big) - f\big(y_0(t), y_n\big(t - \beta z_n(t)\big), \sigma\big) \big| \\
&\qquad + \sup_{t \in [0, 2\pi]} \big| f\big(y_0(t), y_n\big(t - \beta z_n(t)\big), \sigma\big) - f\big(y_0(t), y_0\big(t - \beta z_n(t)\big), \sigma\big) \big| \\
&\qquad + \sup_{t \in [0, 2\pi]} \big| f\big(y_0(t), y_0\big(t - \beta z_n(t)\big), \sigma\big) - f\big(y_0(t), y_0\big(t - \beta z_0(t)\big), \sigma\big) \big| \\
&\qquad + \sup_{t \in [0, 2\pi]} \big| g\big(y_n(t), z_n(t), \sigma\big) - g\big(y_0(t), z_n(t), \sigma\big) \big| \\
&\qquad + \sup_{t \in [0, 2\pi]} \big| g\big(y_0(t), z_n(t), \sigma\big) - g\big(y_0(t), z_0(t), \sigma\big) \big| \\
&\quad = \sup_{t \in [0, 2\pi]} \left| \int_0^1 \partial_1 f\big(y_n(t) + s\big(y_n(t) - y_0(t)\big), y_n\big(t - \beta z_n(t)\big), \sigma\big)\, ds \,\big(y_n(t) - y_0(t)\big) \right| \\
&\qquad + \sup_{t \in [0, 2\pi]} \left| \int_0^1 \partial_2 f\big(y_0(t), y_0\big(t - \beta z_n(t)\big) \right. \\
&\qquad \left. + s\big(y_n\big(t - \beta z_n(t)\big) - y_0\big(t - \beta z_n(t)\big)\big), \sigma\big)\, ds \,\big(y_n(t) - y_0(t)\big) \right| \\
&\qquad + \sup_{t \in [0, 2\pi]} \left| \int_0^1 \partial_2 f\big(y_0(t), y_0\big(t - \beta z_n(t)\big) \right. \\
&\qquad \left. + s\big(y_0\big(t - \beta z_n(t)\big) - y_0\big(t - \beta z_0(t)\big)\big), \sigma\big)\, ds \,\big(y_0\big(t - \beta z_n(t)\big) - y_0\big(t - \beta z_0(t)\big)\big) \right| \\
&\qquad + \sup_{t \in [0, 2\pi]} \left| \int_0^1 \partial_1 g\big(y_0 + s(y_n - y_0), z_n, \sigma\big)\, ds \,\big(y_n(t) - y_0(t)\big) \right| \\
&\qquad + \sup_{t \in [0, 2\pi]} \left| \int_0^1 \partial_2 g\big(y_0, z_0 + s(z_n - z_0), \sigma\big)\, ds \,\big(z_n(t) - z_0(t)\big) \right| \\
&\quad \leqslant \sup_{t \in [0, 2\pi]} \sup_{s \in [0, 1]} \big| \partial_1 f\big(y_n(t) + s\big(y_n(t) - y_0(t)\big), y_n\big(t - \beta z_n(t)\big), \sigma\big) \big| \cdot \| y_n - y_0 \|
\end{aligned}
$$

$$
\begin{aligned}
&+\sup_{t\in[0,2\pi]}\sup_{s\in[0,1]}\Bigl|\partial_2 f\bigl(y_0(t), y_0\bigl(t-\beta z_n(t)\bigr)\\
&+s\bigl(y_n\bigl(t-\beta z_n(t)\bigr)-y_0\bigl(t-\beta z_n(t)\bigr)\bigr),\sigma\bigr)\Bigr|\cdot\|y_n-y_0\|\\
&+\sup_{t\in[0,2\pi]}\sup_{s\in[0,1]}\Bigl|\partial_2 f\bigl(y_0(t), y_0\bigl(t-\beta z_n(t)\bigr)+s\bigl(y_0\bigl(t-\beta z_n(t)\bigr)-y_0\bigl(t-\beta z_0(t)\bigr)\bigr),\sigma\bigr)\Bigr|\\
&\cdot\sup_{t\in[0,2\pi]}\bigl|y_0\bigl(t-\beta z_n(t)\bigr)-y_0\bigl(t-\beta z_0(t)\bigr)\bigr|\\
&+\sup_{t\in[0,2\pi]}\sup_{s\in[0,1]}\bigl|\partial_1 g\bigl(y_0+s(y_n-y_0), z_n,\sigma\bigr)\bigr|\cdot\|y_n-y_0\|\\
&+\sup_{t\in[0,2\pi]}\sup_{s\in[0,1]}\bigl|\partial_2 g\bigl(y_0, z_0+s(z_n-z_0),\sigma\bigr)\bigr|\cdot\|z_n-z_0\|.
\end{aligned}
\tag{3.15}
$$

Since $\lim_{n\to\infty}\|(y_n,z_n)-(y_0,z_0)\|=0$, $\{(y_n(t),z_n(t))\colon\ t\in\mathbb{R},\ n\in\mathbb{N}\}$ is bounded in $\mathbb{R}^{N+1}$ and hence the first two arguments of the partial derivatives in the last inequality of (3.15) are bounded. Then by (S1) we know that there exists a constant $\tilde{L}_0>0$ so that

$$
\begin{cases}
\sup_{t\in[0,2\pi]}\sup_{s\in[0,1]}\Bigl|\partial_1 f\bigl(y_n(t)+s\bigl(y_n(t)-y_0(t)\bigr), y_n\bigl(t-\beta z_n(t)\bigr),\sigma\bigr)\Bigr|\leqslant\tilde{L}_0,\\
\sup_{t\in[0,2\pi]}\sup_{s\in[0,1]}\Bigl|\partial_2 f\bigl(y_0(t), y_0\bigl(t-\beta z_n(t)\bigr)\\
\quad+s\bigl(y_n\bigl(t-\beta z_n(t)\bigr)-y_0\bigl(t-\beta z_n(t)\bigr)\bigr),\sigma\bigr)\Bigr|\leqslant\tilde{L}_0,\\
\sup_{t\in[0,2\pi]}\sup_{s\in[0,1]}\Bigl|\partial_2 f\bigl(y_0(t), y_0\bigl(t-\beta z_n(t)\bigr)\\
\quad+s\bigl(y_0\bigl(t-\beta z_n(t)\bigr)-y_0\bigl(t-\beta z_0(t)\bigr)\bigr),\sigma\bigr)\Bigr|\leqslant\tilde{L}_0,\\
\sup_{t\in[0,2\pi]}\sup_{s\in[0,1]}\bigl|\partial_1 g\bigl(y_0+s(y_n-y_0), z_n,\sigma\bigr)\bigr|\leqslant\tilde{L}_0,\\
\sup_{t\in[0,2\pi]}\sup_{s\in[0,1]}\bigl|\partial_2 g\bigl(y_0, z_0+s(z_n-z_0),\sigma\bigr)\bigr|\leqslant\tilde{L}_0.
\end{cases}
\tag{3.16}
$$

Also, y_0 is a 2π-periodic continuous function and hence y_0 is uniformly continuous on $\mathbb{R}$. Therefore,

$$
\lim_{n\to\infty}\bigl\|y_0\bigl(\cdot-\beta z_n(\cdot)\bigr)-y_0(\cdot-\beta z_0)\bigr\|=0.
\tag{3.17}
$$

Then by (3.15), (3.16) and (3.17), we have

$$
\lim_{n\to\infty}\bigl\|N_0(y_n,z_n,\sigma,\beta)-N_0(y_0,z_0,\sigma,\beta)\bigr\|=0,
$$

which implies that $N_0(\cdot,\sigma,\beta): V\to V$ is continuous. □

Lemma 3.3. *If system* (3.1) *has a nonconstant periodic solution with period $T>0$, then there exists an integer $m\geqslant 1$, $m\in\mathbb{N}$ such that $\pm im2\pi/T$ are characteristic values of the stationary state $(x_\sigma,\tau_\sigma,\sigma)$.*

Proof. Suppose that (x,τ) is a nonconstant T-periodic solution of system (3.1) at σ with $T>0$. Let $\beta=\frac{2\pi}{T}$ and

$$
\bigl(x(t),\tau(t)\bigr)-(x_\sigma,\tau_\sigma)=\bigl(y(\beta t), z(\beta t)\bigr)
$$

for any $t \in \mathbb{R}$. Then $u = (y, z)$ has period 2π and (y, z) is a nonconstant solution of the following system

$$\begin{aligned}\begin{pmatrix}\dot{y}(t)\\ \dot{z}(t)\end{pmatrix} &= \frac{1}{\beta}\begin{bmatrix}\partial_1 f(\sigma) & 0\\ \partial_1 g(\sigma) & \partial_2 g(\sigma)\end{bmatrix}\begin{pmatrix}y(t)\\ z(t)\end{pmatrix} + \frac{1}{\beta}\begin{bmatrix}\partial_2 f(\sigma) & 0\\ 0 & 0\end{bmatrix}\begin{pmatrix}y(t-\beta z_\sigma)\\ z(t-\beta z_\sigma)\end{pmatrix}\\ &= \frac{1}{\beta}\tilde{N}_0(u, \sigma, \beta)(t),\end{aligned} \tag{3.18}$$

where $\tilde{N}_0 : V \times \mathbb{R}^2 \ni (u, \sigma, \beta) \to \tilde{N}_0(u, \sigma, \beta) \in V$ is defined in the formula following (3.7). Let L_0 and K be as in Lemma 3.1, then (3.18) is equivalent to

$$u - (L_0 + K)^{-1}\left(\frac{1}{\beta}\tilde{N}_0(u, \sigma, \beta) + K(u)\right) = 0. \tag{3.19}$$

It is known that the space V has an isotypical direct sum decomposition (see [24], p. 231)

$$V = \overline{\bigoplus_{k=0}^{\infty} V_k},$$

where V_0 is the space of all constant mappings from $\mathbb{R}$ into $\mathbb{R}^{N+1}$, and V_k, $k > 0$, $k \in \mathbb{N}$ is the vector space of all mappings of the form $x \cos k\cdot + y \sin k\cdot : \mathbb{R} \ni t \to x \cos kt + y \sin kt \in \mathbb{R}^{N+1}$, $x, y \in \mathbb{R}^{N+1}$. Note that $u \in V$ is infinitely differentiable. Then u has an uniformly convergent Fourier series (see, e.g., [21], p. 157). That is, for any $k \geqslant 0$, $k \in \mathbb{N}$, there exists $u_k \in V_k$ such that

$$u = \sum_{k=0}^{\infty} u_k. \tag{3.20}$$

We denote by $\tilde{V}_k$ the vector space over the complex numbers which is spanned by $e^{ik\cdot} \cdot \epsilon_j : \mathbb{R} \ni t \to e^{ikt} \cdot \epsilon_j \in \mathbb{C}^{N+1}$, $j = 1, 2, \ldots, N+1$, where $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_{N+1}\}$ denotes the standard basis of $\mathbb{R}^{N+1}$. Then we can define a linear isomorphism $J_0 : V_k \to \tilde{V}_k, k \geqslant 1, k \in \mathbb{N}$, by

$$J_0(x \cos k\cdot + y \sin k\cdot) = \frac{1}{2}\theta^k (x - iy) \tag{3.21}$$

where $x, y \in \mathbb{R}^{N+1}$ and $\theta^k = e^{ik\cdot}$ is the map defined by $\theta^k : \mathbb{R} \ni t \to e^{ikt} \in S^1$.

Let $\Psi(\sigma, \beta) = \mathrm{Id} - (L_0 + K)^{-1}(\frac{1}{\beta} D_u \tilde{N}_0(\cdot, \sigma, \beta) + K)$ and note that N_1 is linear and continuous in u. By (3.19) and (3.20), we have

$$\Psi(\sigma, \beta)(u) = \sum_{k=0}^{\infty} \Psi(\sigma, \beta) u_k = 0. \tag{3.22}$$

Note that $u \in V$ is infinitely differentiable. We can obtain the uniformly convergent Fourier series of $\dot{u} \in V$ through term by term differentiation on both sides of (3.20) with $\dot{u}_k \in V_k$ (see, e.g., [21], p. 165). Therefore, we assume, without loss of generality, that $u_0 = 0$, for otherwise we replace the sequence $\{u_k\}_{k=0}^{+\infty}$ by $\{\dot{u}_k\}_{k=0}^{+\infty}$ with $\dot{u} = \sum_{k=1}^{+\infty} \dot{u}_k$.

Now we show that $\Psi(\sigma,\beta)u_k \in V_k$ for every $k \geqslant 1$, $k \in \mathbb{N}$. Indeed, for every $u_k = x\cos k\cdot + y\sin k\cdot$, $x, y \in \mathbb{R}^{N+1}$, by (3.14) we have

$$\begin{cases} K(u_k) = 0, \\ (L_0 + K)^{-1}(\cos k\cdot) = \dfrac{1}{k}\sin k\cdot, \\ (L_0 + K)^{-1}(\sin k\cdot) = -\dfrac{1}{k}\cos k\cdot\,. \end{cases} \tag{3.23}$$

By linearity of $(L_0 + K)^{-1}$ and (3.23), we obtain

$$\begin{aligned}
&\Psi(\sigma,\beta)(x\cos k\cdot + y\sin k\cdot) \\
&\quad = x\cos k\cdot + y\sin k\cdot - \frac{1}{\beta}(L_0 + K)^{-1}\left\{\begin{bmatrix} \partial_1 f(\sigma) & 0 \\ \partial_1 g(\sigma) & \partial_2 g(\sigma) \end{bmatrix}(x\cos k\cdot + y\sin k\cdot)\right. \\
&\qquad \left. + \begin{bmatrix} \partial_2 f(\sigma) & 0 \\ 0 & 0 \end{bmatrix}\big(x\cos k(\cdot - \beta z_\sigma) + y\sin k(\cdot - \beta z_\sigma)\big)\right\} \\
&\quad = x\cos k\cdot + y\sin k\cdot \\
&\qquad - \frac{1}{\beta}\left\{\begin{bmatrix} \partial_1 f(\sigma) & 0 \\ \partial_1 g(\sigma) & \partial_2 g(\sigma) \end{bmatrix}\big(x(L_0 + K)^{-1}\cos k\cdot + y(L_0 + K)^{-1}\sin k\cdot\big)\right. \\
&\qquad \left. + \begin{bmatrix} \partial_2 f(\sigma) & 0 \\ 0 & 0 \end{bmatrix}\big(x(L_0 + K)^{-1}\cos k(\cdot - \beta z_\sigma) + y(L_0 + K)^{-1}\sin k(\cdot - \beta z_\sigma)\big)\right\} \\
&\quad = x\cos k\cdot + y\sin k\cdot - \frac{1}{k\beta}\left\{\begin{bmatrix} \partial_1 f(\sigma) & 0 \\ \partial_1 g(\sigma) & \partial_2 g(\sigma) \end{bmatrix}(x\sin k\cdot - y\cos k\cdot)\right. \\
&\qquad \left. + \begin{bmatrix} \partial_2 f(\sigma) & 0 \\ 0 & 0 \end{bmatrix}\big(x\sin k(\cdot - \beta z_\sigma) - y\cos k(\cdot - \beta z_\sigma)\big)\right\}.
\end{aligned} \tag{3.24}$$

Note that each term in the last equality of (3.24) is a linear combination of $\cos k\cdot$ and $\sin k\cdot$. Therefore, we have $\Psi(\sigma,\beta)u_k \in V_k$. Then by (3.22) there exists $m \in \mathbb{N}$, $m \geqslant 1$ such that $u_m \neq 0$ and

$$\Psi(\sigma,\beta)u_m = 0. \tag{3.25}$$

We note that every element in $\tilde{V}_k$ can be written as a linear combination of the elements in V_k with complex coefficients. Therefore, we can extend the domain of $\Psi(\sigma,\beta)$ from V_k to $\tilde{V}_k$ using this linearity. We claim that for this extension, we have $\Psi(\sigma,\beta)J_0(u_m) = 0$. To verify this claim, we show that the composition of J_0 and $\Psi(\sigma,\beta)$ is commutative on $V_k, k \geqslant 1, k \in \mathbb{N}$. By the linearity of J_0 and $(L_0 + K)^{-1}$, we only need to show that $J_0(L_0 + K)^{-1} = (L_0 + K)^{-1}J_0$ holds on V_k. Indeed, let $u_k = x\cos k\cdot + y\sin k\cdot \in V_k$. Then it follows from (3.21) and (3.23) that

$$\begin{aligned}
J_0(L_0 + K)^{-1}(x\cos k\cdot + y\sin k\cdot) &= \frac{1}{k}J_0(x\sin k\cdot - y\cos k\cdot) \\
&= -\frac{1}{2k}\theta^k(y + ix)
\end{aligned}$$

and

$$\begin{aligned}(L_0+K)^{-1}J_0(x\cos k\cdot+y\sin k\cdot)&=\frac{1}{2}(L_0+K)^{-1}\theta^k(x-iy)\\&=\frac{1}{2k}(x-iy)(\sin k\cdot-i\cos k\cdot)\\&=-\frac{1}{2k}(y+ix)(\cos k\cdot+i\sin k\cdot)\\&=-\frac{1}{2k}\theta^k(y+ix),\end{aligned}$$

where $\theta^k=e^{ik\cdot}$. This shows that $J_0(L_0+K)^{-1}=(L_0+K)^{-1}J_0$ holds on V_k and hence the composition of J_0 and $\Psi(\sigma,\beta)$ is commutative on V_k, $k\geqslant 1$, $k\in\mathbb{N}$. Therefore, by (3.25) we have

$$J_0\Psi(\sigma,\beta)u_m=\Psi(\sigma,\beta)J_0(u_m)=0. \tag{3.26}$$

Now we denote $\Psi_m(\sigma,\beta)=\Psi(\sigma,\beta)|_{\tilde V_m}$. We have, for $v_m\in\tilde V_m$,

$$\Psi_m(\sigma,\beta)v_m=v_m-\frac{1}{\beta}(L_0+K)^{-1}\left\{\begin{bmatrix}\partial_1 f(\sigma) & 0\\ \partial_1 g(\sigma) & \partial_2 g(\sigma)\end{bmatrix}v_m+\begin{bmatrix}\partial_2 f(\sigma) & 0\\ 0 & 0\end{bmatrix}(v_m)_{\beta z_\sigma}\right\},$$

where $(v_m)_{\beta z_\sigma}=v_m(\cdot-\beta z_\sigma)$. By replacing k, x and y in (3.24) by m, ϵ_j and $i\epsilon_j$, respectively, we have

$$\begin{aligned}&\Psi_m(\sigma,\beta)\big(e^{im\cdot}\epsilon_j\big)\\&\quad=\frac{1}{im\beta}\begin{bmatrix}im\beta\mathrm{Id}-\partial_1 f(\sigma)-\partial_2 f(\sigma)e^{-im\beta z_\sigma} & 0\\ -\partial_1 g(\sigma) & im\beta-\partial_2 g(\sigma)\end{bmatrix}\cdot\big(e^{im\cdot}\epsilon_j\big)\\&\quad=\frac{1}{im\beta}\Delta_{(u(\sigma),\sigma,\beta)}(im\beta)\cdot e^{im\cdot}\epsilon_j\end{aligned} \tag{3.27}$$

for $e^{im\cdot}\epsilon_j\in\tilde V_m$, where the last equality follows from (3.3).

Therefore, the matrix representation $[\Psi_m]$ of $\Psi_m(\sigma,\beta)$ with respect to the ordered basis $\{e^{im\cdot}\epsilon_1, e^{im\cdot}\epsilon_2,\dots,e^{im\cdot}\epsilon_{N+1}\}$ is given by

$$\frac{1}{im\beta}\Delta_{(u(\sigma),\sigma,\beta)}(im\beta).$$

Then by (3.25), we have

$$\frac{1}{im\beta}\Delta_{(u(\sigma),\sigma,\beta)}(im\beta)\bar u_m=0,$$

where $\bar u_m$ is the nonzero coordinate vector of $J(u_m)$ with respect to the ordered basis $\{e^{im\cdot}\epsilon_1,e^{im\cdot}\epsilon_2,\dots,e^{im\cdot}\epsilon_{N+1}\}$ of $\tilde V_m$. Then $im\beta=im2\pi/T$ is a characteristic value of (x_σ,τ_σ) and therefore $\pm im2\pi/T$ are characteristic values of $(x_\sigma,\tau_\sigma,\sigma)$. □

Lemma 3.4. *Assume* (S1)–(S3) *hold. Let* L_0 *and* K *be as in Lemma* 3.1 *and* $\tilde N_0:V\times\mathbb{R}^2\to V$ *be as in (3.7). Define the map* $\tilde{\mathcal{F}}:V\times\mathbb{R}^2\to V$ *by*

$$\tilde{\mathcal{F}}(u,\sigma,\beta):=u-(L_0+K)^{-1}\left[\frac{1}{\beta}\tilde N_0(u,\sigma,\beta)+K(u)\right],$$

where $u=(y,z)$. If $B_M(u_0,\sigma_0,\beta_0;r,\rho)\subseteq V\times\mathbb{R}^2$ is a special neighborhood of $\tilde{\mathcal{F}}$ where $0<\rho<\beta_0$, then there exist $r'\in(0,r]$ such that the neighborhood

$$B_M\big(u_0,\sigma_0,\beta_0;r',\rho\big)=\big\{(u,\sigma,\beta)\colon\ \big\|u-\eta(\sigma)\big\|<r',\ \big|(\sigma,\beta)-(\sigma_0,\beta_0)\big|<\rho\big\}$$

satisfies

$$\dot u(t)\not\equiv\frac{1}{\beta}\begin{pmatrix} f(y(t),y(t-\beta z(t)),\sigma)\\ g(y(t),z(t),\sigma)\end{pmatrix}$$

for $(u,\sigma,\beta)\in\overline{B_M(u_0,\sigma_0,\beta_0;r',\rho)}$ with $u\neq\eta(\sigma)$ and $|(\sigma,\beta)-(\sigma_0,\beta_0)|=\rho$.

Proof. Suppose not, then for any $0<r'\leqslant r$, there exists (u,σ,β) such that $0<\|u-\eta(\sigma)\|<r'$, $|(\sigma,\beta)-(\sigma_0,\beta_0)|=\rho$ and

$$\dot u(t)=\frac{1}{\beta}\begin{pmatrix} f(y(t),y(t-\beta z(t)),\sigma)\\ g(y(t),z(t),\sigma)\end{pmatrix}\quad\text{for } t\in\mathbb{R}. \tag{3.28}$$

Then there exists a sequence of nonconstant periodic solutions $\{(u_k,\sigma_k,\beta_k)=(y_k,z_k,\sigma_k,\beta_k)\}_{k=1}^{\infty}$ of (3.28) such that

$$\lim_{k\to+\infty}\big\|u_k-\eta(\sigma_k)\big\|=0,\qquad \big|(\sigma_k,\beta_k)-(\sigma_0,\beta_0)\big|=\rho, \tag{3.29}$$

and

$$\dot u_k(t)=\frac{1}{\beta_k}\begin{pmatrix} f(y_k(t),y_k(t-\beta_k z_k(t)),\sigma_k)\\ g(y_k(t),z_k(t),\sigma_k)\end{pmatrix}\quad\text{for } t\in\mathbb{R}. \tag{3.30}$$

Note that $0<\rho<\beta_0$ implies that $\beta_k\geqslant\beta_0-\rho>0$ for every $k\in\mathbb{N}$. Also, since the sequence $\{\sigma_k,\beta_k\}_{k=1}^{\infty}$ belongs to a bounded neighborhood of (σ_0,β_0) in $\mathbb{R}^2$, there exists a subsequence, denoted by $\{(\sigma_k,\beta_k)\}_{k=1}^{\infty}$, that converges to (σ^*,β^*) so that $|(\sigma^*,\beta^*)-(\sigma_0,\beta_0)|=\rho$ and $\beta^*>0$. Without loss of generality, we denote this sequence by $\{(\sigma_k,\beta_k)\}_{k=1}^{\infty}$. Then we have

$$\lim_{k\to+\infty}\big\|u_k-\eta(\sigma_k)\big\|=0,\qquad \lim_{k\to+\infty}\big|(\sigma_k,\beta_k)-\big(\sigma^*,\beta^*\big)\big|=0, \tag{3.31}$$

and

$$\dot u_k(t)=\frac{1}{\beta_k}\begin{pmatrix} f(y_k(t),y_k(t-\beta_k z_k(t)),\sigma_k)\\ g(y_k(t),z_k(t),\sigma_k)\end{pmatrix}\quad\text{for } t\in\mathbb{R}. \tag{3.32}$$

Our strategy here is to show that the system

$$\dot v(t)=\frac{1}{\beta^*}\begin{bmatrix}\partial_1 f(\sigma^*) & 0\\ \partial_1 g(\sigma^*) & \partial_2 g(\sigma^*)\end{bmatrix}v(t)+\frac{1}{\beta^*}\begin{bmatrix}\partial_2 f(\sigma^*) & 0\\ 0 & 0\end{bmatrix}v\big(t-\beta^* z_{\sigma^*}\big), \tag{3.33}$$

has a nonconstant periodic solution which contradicts the assumption that $u_0=(y_{\sigma_0},z_{\sigma_0})$ is the only center of (3.7) in $\overline{B_M(u_0,\sigma_0,\beta_0;r,\rho)}$.

By (S1), $f : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \ni (\theta_1, \theta_2, \sigma) \to f(\theta_1, \theta_2, \sigma) \in \mathbb{R}^N$ is C^2 in (θ_1, θ_2) and the map $g : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \ni (\gamma_1, \gamma_2, \sigma) \to g(\gamma_1, \gamma_2, \sigma) \in \mathbb{R}$ is C^2 in (γ_1, γ_2). It follows from the Integral Mean Value Theorem (see [25], p. 341) and (3.32) that

$$\begin{aligned}\dot{u}_k(t) = {} & \frac{1}{\beta_k} \int_0^1 \begin{bmatrix} \partial_1 f_k(\sigma_k, s)(t) & 0 \\ \partial_1 g_k(\sigma_k, s)(t) & \partial_2 g_k(\sigma_k, s)(t) \end{bmatrix} ds \begin{pmatrix} y_k(t) - y_{\sigma_k} \\ z_k(t) - z_{\sigma_k} \end{pmatrix} \\ & + \frac{1}{\beta_k} \int_0^1 \begin{bmatrix} \partial_2 f_k(\sigma_k, s)(t) & 0 \\ 0 & 0 \end{bmatrix} ds \begin{pmatrix} y_k(t - \beta_k z_k(t)) - y_{\sigma_k} \\ z_k(t - \beta_k z_k(t)) - z_{\sigma_k} \end{pmatrix}, \end{aligned} \tag{3.34}$$

where

$$\begin{aligned} \partial_1 f_k(\sigma_k, s)(t) &:= \partial_1 f\big(y_{\sigma_k} + s\big(y_k(t) - y_{\sigma_k}\big), y_{\sigma_k} + s\big(y_k\big(t - z_k(t)\big) - y_{\sigma_k}\big), \sigma_k\big), \\ \partial_2 f_k(\sigma_k, s)(t) &:= \partial_2 f\big(y_{\sigma_k} + s\big(y_k(t) - y_{\sigma_k}\big), y_{\sigma_k} + s\big(y_k\big(t - z_k(t)\big) - y_{\sigma_k}\big), \sigma_k\big), \\ \partial_1 g_k(\sigma_k, s)(t) &:= \partial_1 g\big(y_{\sigma_k} + s\big(y_k(t) - y_{\sigma_k}\big), z_{\sigma_k} + s\big(z_k(t) - z_{\sigma_k}\big), \sigma_k\big), \\ \partial_2 g_k(\sigma_k, s)(t) &:= \partial_2 g\big(y_{\sigma_k} + s\big(y_k(t) - y_{\sigma_k}\big), z_{\sigma_k} + s\big(z_k(t) - z_{\sigma_k}\big), \sigma_k\big). \end{aligned}$$

Put

$$v_k(t) = \frac{u_k(t) - \eta(\sigma_k)}{\|u_k - \eta(\sigma_k)\|}. \tag{3.35}$$

Then we have

$$v_k\big(t - \beta_k z_k(t)\big) = \frac{u_k(t - \beta_k z_k(t)) - \eta(\sigma_k)}{\|u_k - \eta(\sigma_k)\|}. \tag{3.36}$$

By (3.34) and (3.36) we have

$$\begin{aligned}\dot{v}_k(t) = {} & \frac{1}{\beta_k} \int_0^1 \begin{bmatrix} \partial_1 f_k(\sigma_k, s)(t) & 0 \\ \partial_1 g_k(\sigma_k, s)(t) & \partial_2 g_k(\sigma_k, s)(t) \end{bmatrix} ds\, v_k(t) \\ & + \frac{1}{\beta_k} \int_0^1 \begin{bmatrix} \partial_2 f_k(\sigma_k, s)(t) & 0 \\ 0 & 0 \end{bmatrix} ds\, v_k\big(t - \beta_k z_k(t)\big). \end{aligned} \tag{3.37}$$

We claim that there exists a convergent subsequence of $\{v_k\}_{k=1}^{+\infty}$. Indeed, by (3.29), we know that $\{z_k, \beta_k\}_{k=1}^{+\infty}$ is uniformly bounded in $C(\mathbb{R}; \mathbb{R}) \times \mathbb{R}$ and hence $\lim_{t \to +\infty} [t - \beta_k z_k(t)] = +\infty$. Then by (3.35) and (3.36), we have

$$\|v_k\| = 1, \qquad \big\| v_k\big(\cdot - \beta_k z_k(\cdot)\big) \big\| = 1.$$

Recall that $\partial_i f(\sigma^*)$ and $\partial_i g(\sigma^*)$, $i = 1, 2$, are defined in (3.1). By (3.31), we know that

$$\big(y_{\sigma_k} + s\big(y_k(t) - y_{\sigma_k}\big), y_{\sigma_k} + s\big(y_k\big(t - z_k(t)\big) - y_{\sigma_k}\big), \sigma_k\big)$$

converges to the stationary state $(x_{\sigma^*}, \tau_{\sigma^*}, \sigma^*)$ in $C(\mathbb{R};\mathbb{R}) \times \mathbb{R}$ uniformly for all $s \in [0,1]$. By (S1) we know that $f(\theta_1,\theta_2,\sigma)$ is C^2 in (θ_1,θ_2) and $\partial_1 f(\theta_1,\theta_2,\sigma)$ is C^1 in σ. Also, by (3.29), the sequence $\{u_k,\beta_k,\sigma_k\}_{k=1}^{+\infty}$ is uniformly bounded in $C(\mathbb{R};\mathbb{R}^{N+1}) \times \mathbb{R}^2$. Then there exists a constant $\tilde{L}_1 > 0$ so that

$$\begin{aligned}
&\left|\partial_1 f_k(\sigma_k,s)(t) - \partial_1 f(\sigma_0)\right| \\
&\quad \leqslant \tilde{L}_1 \left|\left(y_{\sigma_k} + s\left(y_k(t) - y_{\sigma_k}\right), y_{\sigma_k} + s\left(y_k\left(t - z_k(t)\right) - y_{\sigma_k}\right), \sigma_k\right) - \left(x_{\sigma^*}, \tau_{\sigma^*}, \sigma^*\right)\right|,
\end{aligned}$$

for all $t \in \mathbb{R}$, $k \in \mathbb{N}$ and $s \in [0,1]$. Therefore, we have $\lim_{k\to+\infty} \|\partial_1 f_k(\sigma_k,s) - \partial_1 f(\sigma^*)\| = 0$ uniformly for $s \in [0,1]$. By the same argument we obtain that

$$\begin{cases}
\lim\limits_{k\to+\infty} \left\|\partial_1 f_k(\sigma_k,s) - \partial_1 f\left(\sigma^*\right)\right\| = 0,\\
\lim\limits_{k\to+\infty} \left\|\partial_2 f_k(\sigma_k,s) - \partial_2 f\left(\sigma^*\right)\right\| = 0,\\
\lim\limits_{k\to+\infty} \left\|\partial_1 g_k(\sigma_k,s) - \partial_1 g\left(\sigma^*\right)\right\| = 0,\\
\lim\limits_{k\to+\infty} \left\|\partial_2 g_k(\sigma_k,s) - \partial_2 g\left(\sigma^*\right)\right\| = 0,
\end{cases} \tag{3.38}$$

uniformly for $s \in [0,1]$. It is clear from (3.38) that $\|\partial_1 f_k(\sigma_k,s)\|$, $\|\partial_2 f_k(\sigma_k,s)\|$, $\|\partial_1 g_k(\sigma_k,s)\|$ and $\|\partial_2 g_k(\sigma_k,s)\|$ are all uniformly bounded for all $k \in \mathbb{N}$ and $s \in [0,1]$. Then it follows from (3.37) that there exists a constant $\tilde{L}_2 > 0$ such that $\|\dot{v}_k\| < \tilde{L}_2$ for any $k \in \mathbb{N}$. By the Arzela–Ascoli Theorem, there exists a convergent subsequence $\{v_{k_j}\}_{j=1}^{+\infty}$ of $\{v_k\}_{k=1}^{+\infty}$. That is, there exists $v^* \in \{v \in V\colon \|v\| = 1\}$ such that

$$\lim_{j\to+\infty} \left\|v_{k_j} - v^*\right\| = 0. \tag{3.39}$$

By the Integral Mean Value Theorem, we have

$$\begin{aligned}
&\left|v_{k_j}\left(t - \beta_{k_j} z_{k_j}(t)\right) - v_{k_j}\left(t - \beta^* z_{\sigma^*}\right)\right| \\
&\quad = \left|\int_0^1 \dot{v}_{k_j}\left(t - \theta\left(\beta_{k_j} z_{k_j}(t) - \beta^* z_{\sigma^*}\right)\right) d\theta \left(\beta_{k_j} z_{k_j}(t) - \beta^* z_{\sigma^*}\right)\right| \\
&\quad \leqslant \|\dot{v}_{k_j}\| \cdot \left|\beta_{k_j} z_{k_j}(t) - \beta^* z_{\sigma^*}\right| \\
&\quad \leqslant \tilde{L}_2\left(\beta_{k_j}\left|z_{k_j}(t) - z_{\sigma^*}\right| + \left|\beta_{k_j} - \beta^*\right| z_{\sigma^*}\right).
\end{aligned} \tag{3.40}$$

By (3.31) and (3.40) we have

$$\lim_{j\to+\infty} \left\|v_{k_j}\left(\cdot - \beta_{k_j} z_{k_j}(\cdot)\right) - v_{k_j}\left(\cdot - \beta^* z_{\sigma^*}\right)\right\| = 0. \tag{3.41}$$

Therefore, it follows from (3.39) and (3.41) that

$$\lim_{j\to+\infty} \left\|v_{k_j}\left(\cdot - \beta_{k_j} z_{k_j}(\cdot)\right) - v^*\left(\cdot - \beta^* z_{\sigma^*}\right)\right\| = 0. \tag{3.42}$$

It follows from (3.31), (3.38), (3.39) and (3.42) that the right-hand side of (3.37) converges uniformly to the right-hand side of (3.33). Therefore, v^* is differentiable and we have

$$\lim_{k\to+\infty}\left|\dot v_k(t)-\dot v^*(t)\right|=0,$$

and

$$\dot v^*(t)=\frac{1}{\beta^*}\begin{bmatrix}\partial_1 f(\sigma^*) & 0\\ \partial_1 g(\sigma^*) & \partial_2 g(\sigma^*)\end{bmatrix}v^*(t)+\frac{1}{\beta^*}\begin{bmatrix}\partial_2 f(\sigma^*) & 0\\ 0 & 0\end{bmatrix}v^*\bigl(t-\beta^* z_{\sigma^*}\bigr). \tag{3.43}$$

Since by (S3) the matrix

$$\begin{bmatrix}\partial_1 f(\sigma^*)+\partial_2 f(\sigma^*) & 0\\ \partial_1 g(\sigma^*) & \partial_2 g(\sigma^*)\end{bmatrix}$$

is nonsingular, $v=0$ is the only constant solution of (3.43). Also, we have $v^*\in\{v\in V\colon\ \|v\|=1\}$, $\|v^*\|\neq 0$. Therefore, $(v^*(t),\sigma^*,\beta^*)$ is a nonconstant periodic solution of the linear equation (3.43). Then by Lemma 3.3 $(\eta(\sigma^*),\sigma^*,\beta^*)$ is also a center of (3.7) in $\overline{B_M(u_0,\sigma_0,\beta_0;r,\rho)}$. This contradicts the assumption that $B_M(u_0,\sigma_0,\beta_0;r,\rho)$ is a special neighborhood of (3.6). This completes the proof. □

As a preparation for the proof of our local Hopf bifurcation theorem, we need the following

Lemma 3.5. *Assume* (S1)–(S3) *hold. Let* L_0, K, $\tilde N_0$, $\tilde{\mathcal F}$ *be as in Lemma* 3.4 *and* $N_0: V\times\mathbb{R}^2\to V$ *be as in* (3.6). *Define the map* $\mathcal F: V\times\mathbb{R}^2\to V$ *by*

$$\mathcal F(u,\sigma,\beta):=u-(L_0+K)^{-1}\left[\frac{1}{\beta}N_0(u,\sigma,\beta)+K(u)\right].$$

If $\mathcal U=B_M(u_0,\sigma_0,\beta_0;r,\rho)\subseteq V\times\mathbb{R}^2$ *is a special neighborhood of* $\tilde{\mathcal F}$ *with* $0<\rho<\beta_0$*, then there exists* $r'\in(0,r]$ *such that* $\mathcal F_\theta=(\mathcal F,\theta)$ *and* $\tilde{\mathcal F}_\theta=(\tilde{\mathcal F},\theta)$ *are homotopic on* $\overline{B_M(u_0,\sigma_0,\beta_0;r',\rho)}$*, where* θ *is a completing function defined on* $\overline{B_M(u_0,\sigma_0,\beta_0;r',\rho)}$.

Proof. Since $\mathcal U=B_M(u_0,\sigma_0,\beta_0;r,\rho)\subseteq V\times\mathbb{R}^2$ is a special neighborhood of $\tilde{\mathcal F}$ with $0<\rho<\beta_0$, then by Lemma 3.4, both $\mathcal F_\theta=(\mathcal F,\theta)$ and $\tilde{\mathcal F}_\theta=(\mathcal F,\theta)$ are $\mathcal U$-admissible.

Suppose that the conclusion is not true, then for any $r'\in(0,r]$, $\mathcal F_\theta=(\mathcal F,\theta)$ and $\tilde{\mathcal F}_\theta=(\mathcal F,\theta)$ are not homotopic on $\overline{B_M(u_0,\sigma_0,\beta_0;r',\rho)}$. That is, any homotopy map between $\mathcal F_\theta$ and $\tilde{\mathcal F}_\theta$ has a zero on the boundary of $\overline{B_M(u_0,\sigma_0,\beta_0;r',\rho)}$. In particular, the linear homotopy $h(\cdot,\alpha):=\alpha\mathcal F_\theta+(1-\alpha)\tilde{\mathcal F}_\theta=(\alpha\mathcal F+(1-\alpha)\tilde{\mathcal F},\theta)$ has a zero on the boundary of $\overline{B_M(u_0,\sigma_0,\beta_0;r',\rho)}$, where $\alpha\in[0,1]$.

Note that $\theta(u,\sigma,\beta)<0$ if $\|u-\eta(\sigma)\|=r'$. Then, there exist (u,σ,β) and $\alpha\in[0,1]$ such that $\|u-\eta(\sigma)\|<r'$, $|(\sigma,\beta)-(\sigma_0,\beta_0)|=\rho$ and

$$H(u,\sigma,\beta,\alpha):=\alpha\mathcal F+(1-\alpha)\tilde{\mathcal F}=0. \tag{3.44}$$

Since $r'>0$ is arbitrary in the interval $(0,r]$, there exists a nonconstant sequence $\{(y_k,z_k,\sigma_k,\beta_k,\alpha_k)\}_{k=1}^{\infty}$ of solutions of (3.44) such that

$$\lim_{k\to+\infty}\bigl\|u_k-\eta(\sigma_k)\bigr\|=0,\qquad \bigl|(\sigma_k,\beta_k)-(\sigma_0,\beta_0)\bigr|=\rho,\quad 0\leqslant\alpha_k\leqslant 1, \tag{3.45}$$

and

$$H(u_k,\sigma_k,\beta_k,\alpha_k)=0,\quad \text{for all } k\in\mathbb{N}. \tag{3.46}$$

Note that $0 < \rho < \beta_0$ implies that $\beta_k \geqslant \beta_0 - \rho > 0$ for any $k \in \mathbb{N}$. From (3.45) we know that $\{(\sigma_k, \beta_k, \alpha_k)\}_{k=1}^{\infty}$ belongs to a compact subset of $\mathbb{R}^3$. Therefore, there exist a convergent subsequence, denoted still by $\{(\sigma_k, \beta_k, \alpha_k)\}_{k=1}^{\infty}$ without loss of generality, and $(\sigma^*, \beta^*, \alpha^*) \in \mathbb{R}^3$ such that $\beta^* \geqslant \beta_0 - \rho > 0$, $\alpha^* \in [0, 1]$ and

$$\lim_{k \to +\infty} \left|(\sigma_k, \beta_k, \alpha_k) - \left(\sigma^*, \beta^*, \alpha^*\right)\right| = 0. \tag{3.47}$$

Similarly to the proof of Lemma 3.4, we show that the system

$$\dot{v}(t) = \frac{1}{\beta^*} \begin{bmatrix} \partial_1 f(\sigma^*) & 0 \\ \partial_1 g(\sigma^*) & \partial_2 g(\sigma^*) \end{bmatrix} v(t) + \frac{1}{\beta^*} \begin{bmatrix} \partial_2 f(\sigma^*) & 0 \\ 0 & 0 \end{bmatrix} v\left(t - \beta^* z_{\sigma^*}\right) \tag{3.48}$$

with $\partial_i f(\sigma^*), \partial_i g(\sigma^*), i = 1, 2$, defined after (3.1), has a nonconstant periodic solution which contradicts the assumption that $B_M(u_0, \sigma_0, \beta_0; r, \rho)$ is a special neighborhood which contains an isolated center of (3.7).

By (3.46), we know that the subsequence $\{(y_k, z_k, \sigma_k, \beta_k, \alpha_k)\}_{k=1}^{\infty}$ satisfies

$$H(u_k, \sigma_k, \beta_k, \alpha_k) = 0. \tag{3.49}$$

By (S1), $f : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \ni (\theta_1, \theta_2, \sigma) \to f(\theta_1, \theta_2, \sigma) \in \mathbb{R}^N$ is C^2 in (θ_1, θ_2) and the map $g : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \ni (\gamma_1, \gamma_2, \sigma) \to g(\gamma_1, \gamma_2, \sigma) \in \mathbb{R}$ is C^2 in (γ_1, γ_2). Then it follows from the Integral Mean Value Theorem and from (3.49) that

$$\begin{aligned}
\dot{u}_k(t) = {} & \frac{\alpha_k}{\beta_k} \int_0^1 \begin{bmatrix} \partial_1 f_k(\sigma_k, s) & 0 \\ \partial_1 g_k(\sigma_k, s) & \partial_2 g_k(\sigma_k, s) \end{bmatrix} ds \begin{pmatrix} y_k(t) - y_{\sigma_k} \\ z_k(t) - z_{\sigma_k} \end{pmatrix} \\
& + \frac{\alpha_k}{\beta_k} \int_0^1 \begin{bmatrix} \partial_2 f_k(\sigma_k, s) & 0 \\ 0 & 0 \end{bmatrix} ds \begin{pmatrix} y_k(t - \beta_k z_k(t)) - y_{\sigma_k} \\ z_k(t - \beta_k z_k(t)) - z_{\sigma_k} \end{pmatrix} \\
& + \frac{1 - \alpha_k}{\beta_k} \int_0^1 \begin{bmatrix} \partial_1 f_k(\sigma_k, s) & 0 \\ \partial_1 g_k(\sigma_k, s) & \partial_2 g_k(\sigma_k, s) \end{bmatrix} ds \begin{pmatrix} y_k(t) - y_{\sigma_k} \\ z_k(t) - z_{\sigma_k} \end{pmatrix} \\
& + \frac{1 - \alpha_k}{\beta_k} \int_0^1 \begin{bmatrix} \partial_2 f_k(\sigma_k, s) & 0 \\ 0 & 0 \end{bmatrix} ds \begin{pmatrix} y_k(t - \beta_k z_{\sigma_k}) - y_{\sigma_k} \\ z_k(t - \beta_k z_{\sigma_k}) - z_{\sigma_k} \end{pmatrix},
\end{aligned} \tag{3.50}$$

where

$$\begin{aligned}
\partial_1 f_k(\sigma_k, s) &:= \partial_1 f\left(y_{\sigma_k} + s\left(y_k(t) - y_{\sigma_k}\right), y_{\sigma_k} + s\left(y_k\left(t - \beta z_k(t)\right) - y_{\sigma_k}\right), \sigma_k\right), \\
\partial_2 f_k(\sigma_k, s) &:= \partial_2 f\left(y_{\sigma_k} + s\left(y_k(t) - y_{\sigma_k}\right), y_{\sigma_k} + s\left(y_k\left(t - \beta z_k(t)\right) - y_{\sigma_k}\right), \sigma_k\right), \\
\partial_1 g_k(\sigma_k, s) &:= \partial_1 g\left(y_{\sigma_k} + s\left(y_k(t) - y_{\sigma_k}\right), z_{\sigma_k} + s\left(z_k(t) - z_{\sigma_k}\right), \sigma_k\right), \\
\partial_2 g_k(\sigma_k, s) &:= \partial_2 g\left(y_{\sigma_k} + s\left(y_k(t) - y_{\sigma_k}\right), z_{\sigma_k} + s\left(z_k(t) - z_{\sigma_k}\right), \sigma_k\right).
\end{aligned}$$

Put

$$v_k(t) = \frac{u_k(t) - \eta(\sigma_k)}{\|u_k - \eta(\sigma_k)\|}. \tag{3.51}$$

Then we have

$$v_k\big(t - \beta_k z_k(t)\big) = \frac{u_k(t - \beta_k z_k(t)) - \eta(\sigma_k)}{\|u_k - \eta(\sigma_k)\|}. \tag{3.52}$$

By (3.50) and (3.52), we have

$$\begin{aligned}
\dot v_k(t) = {} & \frac{\alpha_k}{\beta_k} \int_0^1 \begin{bmatrix} \partial_1 f_k(\sigma_k, s) & 0 \\ \partial_1 g_k(\sigma_k, s) & \partial_2 g_k(\sigma_k, s) \end{bmatrix} ds\, v_k(t) \\
& + \frac{\alpha_k}{\beta_k} \int_0^1 \begin{bmatrix} \partial_2 f_k(\sigma_k, s) & 0 \\ 0 & 0 \end{bmatrix} ds\, v_k(t - \beta_k z_{\sigma_k}) \\
& + \frac{1-\alpha_k}{\beta_k} \int_0^1 \begin{bmatrix} \partial_1 f_k(\sigma_k, s) & 0 \\ \partial_1 g_k(\sigma_k, s) & \partial_2 g_k(\sigma_k, s) \end{bmatrix} ds\, v_k(t) \\
& + \frac{1-\alpha_k}{\beta_k} \int_0^1 \begin{bmatrix} \partial_2 f_k(\sigma_k, s) & 0 \\ 0 & 0 \end{bmatrix} ds\, v_k(t - \beta_k z_{\sigma_k}).
\end{aligned} \tag{3.53}$$

We claim that there exists a convergent subsequence of $\{v_k\}_{k=1}^{+\infty}$. Indeed, by (3.45) we know that $\{z_k, \beta_k\}_{k=1}^{+\infty}$ is uniformly bounded in $C(\mathbb{R}; \mathbb{R}) \times \mathbb{R}$. Therefore we have

$$\lim_{t \to +\infty} \big[t - \beta_k z_k(t)\big] = +\infty. \tag{3.54}$$

By (3.51), (3.52) and (3.54), we have $\|v_k\| = 1$, $\|v_k(\cdot - \beta_k z_k)\| = 1$. Note that by (S1) and (3.47) and by an argument similar yielding (3.38), we know that

$$\begin{cases}
\lim\limits_{k \to +\infty} \big\|\partial_1 f_k(\sigma_k, s) - \partial_1 f\big(\sigma^*\big)\big\| = 0, \\
\lim\limits_{k \to +\infty} \big\|\partial_2 f_k(\sigma_k, s) - \partial_2 f\big(\sigma^*\big)\big\| = 0, \\
\lim\limits_{k \to +\infty} \big\|\partial_1 g_k(\sigma_k, s) - \partial_1 g\big(\sigma^*\big)\big\| = 0, \\
\lim\limits_{k \to +\infty} \big\|\partial_2 g_k(\sigma_k, s) - \partial_2 g\big(\sigma^*\big)\big\| = 0,
\end{cases} \tag{3.55}$$

uniformly for $s \in [0,1]$. It is clear from (3.55) that $\|\partial_1 f_k(\sigma_k, s)\|$, $\|\partial_2 f_k(\sigma_k, s)\|$, $\|\partial_1 g_k(\sigma_k, s)\|$ and $\|\partial_2 g_k(\sigma_k, s)\|$ are all uniformly bounded for any $k \in \mathbb{N}$ and $s \in [0,1]$. It follows from (3.53) that there exists $\tilde L_3 > 0$ such that $\|\dot v_k\| < \tilde L_3$ for any $k \in \mathbb{N}$. By the Arzela–Ascoli Theorem, there exists a convergent subsequence $\{v_{k_j}\}_{j=1}^{+\infty}$ of $\{v_k\}_{k=1}^{+\infty}$. That is, there exists $v^* \in \{v \in V\colon \|v\| = 1\}$ such that

$$\lim_{j \to +\infty} \big\|v_{k_j} - v^*\big\| = 0. \tag{3.56}$$

By the Integral Mean Value Theorem, we have, for all $t \in \mathbb{R}$,

$$\begin{aligned}
&\left|v_{k_j}\big(t-\beta_{k_j}z_{k_j}(t)\big)-v_{k_j}\big(t-\beta^* z_{\sigma^*}\big)\right| \\
&\quad=\left|\int_0^1 \dot v_{k_j}\big(t-\beta^* z_{\sigma^*}-\theta\big(\beta_{k_j}z_{k_j}(t)-\beta^* z_{\sigma^*}\big)\big)\,d\theta\big(\beta_{k_j}z_{k_j}(t)-\beta^* z_{\sigma^*}\big)\right| \\
&\quad\leqslant \|\dot v_{k_j}\|\cdot\left|\beta_{k_j}z_{k_j}(t)-\beta^* z_{\sigma^*}\right| \\
&\quad\leqslant \tilde L_3\big(\beta_{k_j}\left|z_{k_j}(t)-z_{\sigma^*}\right|+\left|\beta_{k_j}-\beta^*\right|z_{\sigma^*}\big).
\end{aligned}\tag{3.57}$$

Then by (3.47) and (3.57) we have

$$\lim_{j\to+\infty}\left\|v_{k_j}\big(\cdot-\beta_{k_j}z_{k_j}(\cdot)\big)-v_{k_j}\big(\cdot-\beta^* z_{\sigma^*}\big)\right\|=0.\tag{3.58}$$

From (3.56) and (3.58) we have

$$\lim_{j\to+\infty}\left\|v_{k_j}\big(\cdot-\beta_{k_j}z_{k_j}(\cdot)\big)-v^*\big(\cdot-\beta^* z_{\sigma^*}\big)\right\|=0.\tag{3.59}$$

It follows from (3.47), (3.55), (3.56) and (3.59) that the right-hand side of (3.53) converges uniformly to the right-hand side of (3.48). Therefore,

$$\lim_{j\to+\infty}\left|\dot v_{k_j}(t)-\dot v^*(t)\right|=0\tag{3.60}$$

and

$$\dot v^*(t)=\frac{1}{\beta^*}\begin{bmatrix}\partial_1 f(\sigma^*) & 0\\ \partial_1 g(\sigma^*) & \partial_2 g(\sigma^*)\end{bmatrix}v^*(t)+\frac{1}{\beta^*}\begin{bmatrix}\partial_2 f(\sigma^*) & 0\\ 0 & 0\end{bmatrix}v^*\big(t-\beta^*\tau_{\sigma^*}\big).\tag{3.61}$$

Since $v^*\in\{v\colon \|v\|=1\}$, $\|v^*\|\neq 0$ and the matrix

$$\begin{bmatrix}\partial_1 f(\sigma^*)+\partial_2 f(\sigma^*) & 0\\ \partial_1 g(\sigma^*) & \partial_2 g(\sigma^*)\end{bmatrix}$$

is nonsingular, v^* is a nonconstant periodic solution of (3.61). By Lemma 3.3 $(\eta(\sigma^*),\sigma^*,\beta^*)$ is also a center of (3.7) in $\overline{B_M(u_0,\sigma_0,\beta_0;r,\rho)}$. This contradicts the assumption that $B_M(u_0,\sigma_0,\beta_0;r,\rho)$ is a special neighborhood of (3.7) which contains only one center (u_0,σ_0,β_0). This completes the proof. □

Now we are able to state and prove our local Hopf bifurcation theorem.

Theorem 3.6. *Assume* (S1)–(S3) *hold. Let* $(x_{\sigma_0},\tau_{\sigma_0},\sigma_0)$ *be an isolated center of system* (3.1)*. If the crossing number defined by* (3.5) *satisfies*

$$\gamma(x_{\sigma_0},\tau_{\sigma_0},\sigma_0,\beta_0)\neq 0,$$

then there exists a bifurcation of nonconstant periodic solutions of (1.1) *near* $(x_{\sigma_0},\tau_{\sigma_0},\sigma_0)$*. More precisely, there exists a sequence* $\{(x_n,\tau_n,\sigma_n,\beta_n)\}$ *such that* $\sigma_n\to\sigma_0$*,* $\beta_n\to\beta_0$ *as* $n\to\infty$*, and* $\lim_{n\to\infty}\|x_n-x_{\sigma_0}\|=0$*,* $\lim_{n\to\infty}\|\tau_n-\tau_{\sigma_0}\|=0$*, where*

$$(x_n, \tau_n, \sigma_n) \in C\big(\mathbb{R}; \mathbb{R}^{N+1}\big) \times \mathbb{R}$$

is a nonconstant $2\pi/\beta_n$-periodic solution of system (1.1)*.*

Proof. Suppose (x, τ) is a $2\pi/\beta$-periodic solution of system (1.1) with $\beta > 0$. Let $(x(t), \tau(t)) = (y(\beta t), z(\beta t))$. Then system (1.1) is transformed to

$$\begin{cases} \dot{y}(t) = \dfrac{1}{\beta} f\big(y(t), y\big(t - \beta z(t)\big), \sigma\big), \\ \dot{z}(t) = \dfrac{1}{\beta} g\big(y(t), z(t), \sigma\big). \end{cases} \tag{3.62}$$

Then (x, τ) is a $2\pi/\beta$-periodic solution of system (1.1) if and only if (y, z) is a 2π-periodic solution of system (3.62).

Let $V = C_{2\pi}(\mathbb{R}; \mathbb{R}^{N+1})$. S^1 acts on V by argument shift. Namely, for any $\xi = e^{i\nu} \in S^1$, $u \in V$, $(\xi u)(t) := u(t + \nu)$. The idea of the proof in the sequel is to verify all the conditions for applying Theorem 2.4.

Recall that δ and ε are defined before (3.5). Let $\mathscr{D}(\sigma_0, \beta_0) = (\sigma_0 - \delta, \sigma_0 + \delta) \times (\beta_0 - \varepsilon, \beta_0 + \varepsilon)$ and define the maps

$$L_0 u(t) := \begin{pmatrix} \dot{y}(t) \\ \dot{z}(t) \end{pmatrix}, \quad u \in C^1_{2\pi}\big(\mathbb{R}; \mathbb{R}^{N+1}\big),$$

$$N_0(u, \sigma, \beta)(t) := \begin{pmatrix} f(y(t), y(t - \beta z(t)), \sigma) \\ g(y(t), z(t), \sigma) \end{pmatrix}, \quad u \in V,$$

$$\tilde{N}_0(u, \sigma, \beta)(t) := \begin{pmatrix} \partial_1 f(\sigma)(y(t) - y_\sigma) + \partial_2 f(\sigma)(y(t - \beta z_\sigma) - y_\sigma) \\ \partial_1 g(\sigma)(y(t) - y_\sigma) + \partial_2 g(\sigma)(z(t) - z_\sigma) \end{pmatrix}, \quad u \in V,$$

where $u = (y, z)$, $(\sigma, \beta) \in \mathscr{D}(\sigma_0, \beta_0)$ and $t \in \mathbb{R}$, and $\eta(\sigma) = (y_\sigma, z_\sigma)$ is the stationary point of the system. The space V is a Banach representation of the group $G = S^1$. Define the operator $K : V \to \mathbb{R}^{N+1}$ by

$$K(u) := \frac{1}{2\pi} \int_0^{2\pi} u(t)\, dt, \quad u \in V.$$

By Lemma 3.1, the operator $L_0 + K : C^1_{2\pi}(\mathbb{R}; \mathbb{R}^{N+1}) \to V$ has a compact inverse $(L_0 + K)^{-1} : V \to V$. Then, finding a $2\pi/\beta$-periodic solution for the system (1.1) is equivalent to finding a solution of the following fixed point problem:

$$u = (L_0 + K)^{-1}\left[\frac{1}{\beta} N_0(u, \sigma, \beta) + K(u)\right], \tag{3.63}$$

where $(u, \sigma, \beta) \in V \times \mathbb{R} \times (0, +\infty)$.

By (S1) we know that the linear operator $\tilde{N}_0$ is continuous. By Lemma 3.2, we know that $N_0(\cdot, \sigma, \beta) : V \to V$ is continuous. Moreover, by Lemma 3.1 the operator $(L_0 + K)^{-1} : V \to V$ is compact and hence $(L_0 + K)^{-1} \circ (\frac{1}{\beta} N_0(\cdot, \alpha, \beta) + K) : V \to V$ and $(L_0 + K)^{-1} \circ (\frac{1}{\beta} \tilde{N}_0(\cdot, \alpha, \beta) + K) : V \to V$ are completely continuous and hence are condensing maps. That is, (A2) and (A4) are satisfied.

In summary, we can define the following maps $\mathcal{F} : V \times \mathbb{R} \times (0, +\infty) \to V$ and $\tilde{\mathcal{F}} : V \times \mathbb{R} \times (0, +\infty) \to V$ by

$$\mathcal{F}(u,\sigma,\beta) := u - (L_0+K)^{-1}\left[\frac{1}{\beta}N_0(u,\sigma,\beta)+K(u)\right],$$

$$\tilde{\mathcal{F}}(u,\sigma,\beta) := u - (L_0+K)^{-1}\left[\frac{1}{\beta}\tilde{N}_0(u,\sigma,\beta)+K(u)\right],$$

which are equivariant condensing fields. Finding a $2\pi/\beta$-periodic solution of system (1.1) is equivalent to finding the solution of the problem

$$\mathcal{F}(u,\sigma,\beta)=0, \quad (u,\sigma,\beta)\in V\times\mathbb{R}\times(0,+\infty).$$

Since $(u_0,\sigma_0):=(x_{\sigma_0},\tau_{\sigma_0},\sigma_0)$ is an isolated center of system (3.1) with a purely imaginary characteristic value $i\beta_0$, $\beta_0>0$, $(u_0,\sigma_0,\beta_0)\in V\times\mathbb{R}\times(0,+\infty)$ is an isolated V-singular point of $\tilde{\mathcal{F}}$. One can define the following two-dimensional submanifold $M\subset V^G\times\mathbb{R}\times(0,+\infty)$ by

$$M:=\big\{\big(\eta(\sigma),\sigma,\beta\big)\colon \sigma\in(\sigma_0-\delta,\sigma_0+\delta),\ \beta\in(\beta_0-\varepsilon,\beta_0+\varepsilon)\big\}$$

such that the point $(\eta(\sigma_0),\sigma_0,\beta_0)=(u_0,\sigma_0,\beta_0)$ is the only V-singular point of $\tilde{\mathcal{F}}$ in M. M is the set of trivial solutions to the system (3.1) and satisfies the assumption (A3).

Moreover, $(u_0,\sigma_0,\beta_0)\in V\times\mathbb{R}\times(0,+\infty)$ is an isolated V-singular point of $\tilde{\mathcal{F}}$. That is, for $\rho>0$ sufficiently small, the linear operator $D_u\tilde{\mathcal{F}}(\eta(\sigma),\sigma,\beta):V\to V$ with $|(\sigma,\beta)-(\sigma_0,\beta_0)|<\rho$, is not an isomorphism only if $(\sigma,\beta)=(\sigma_0,\beta_0)$. Then, by the Implicit Function Theorem, there exists $r>0$ such that for all $(u,\sigma,\beta)\in V\times\mathbb{R}\times(0,+\infty)$ with $|(\sigma,\beta)-(\sigma_0,\beta_0)|=\rho$ and $0<\|u-\eta(\sigma)\|\leqslant r$, we have $\tilde{\mathcal{F}}(u,\sigma,\beta)\neq 0$. Then the set $B_M(u_0,\sigma_0,\beta_0;r,\rho)$ defined by

$$\big\{(u,\sigma,\beta)\in V\times\mathbb{R}\times(0,+\infty);\ \big|(\sigma,\beta)-(\sigma_0,\beta_0)\big|<\rho,\ \big\|u-\eta(\sigma)\big\|<r\big\}$$

is a special neighborhood for $\tilde{\mathcal{F}}$.

By Lemma 3.4, there exists a special neighborhood $\mathcal{U}=B_M(u_0,\sigma_0,\beta_0;r',\rho)$ such that $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are nonzero for $(u,\sigma,\beta)\in\overline{B_M(u_0,\sigma_0,\beta_0;r',\rho)}$ with $u\neq\eta(\sigma)$ and $|(\sigma,\beta)-(\sigma_0,\beta_0)|=\rho$. That is, (A5) is satisfied.

Let θ be a completing function on $\mathcal{U}$. It follows from Lemma 3.5 that $(\mathcal{F},\theta)$ is homotopic to $(\tilde{\mathcal{F}},\theta)$ on $\mathcal{U}$.

It is known that V has the following isotypical direct sum decomposition

$$V=\overline{\bigoplus_{k=0}^{\infty}V_k},$$

where V_0 is the space of all constant mappings from $\mathbb{R}$ into $\mathbb{R}^{N+1}$, and V_k with $k>0$, $k\in\mathbb{N}$ is the vector space of all mappings of the form $x\cos k\cdot+y\sin k\cdot:\mathbb{R}\ni t\to x\cos kt+y\sin kt\in\mathbb{R}^{N+1}$, $x,y\in\mathbb{R}^{N+1}$. It is clear that V_k, $k>0$, $k\in\mathbb{N}$, are finite dimensional. Then, (A1) is satisfied.

For $(\sigma,\beta)\in\mathscr{D}(\sigma_0,\beta_0)$, we denote by $\Psi(\sigma,\beta)$ the map $D_u\tilde{\mathcal{F}}(u(\sigma),\sigma,\beta):V\to V$. By (3.24), we know that $\Psi(\sigma,\beta)(V_k)\subset V_k$ for all $k=0,1,2,\ldots$. Therefore, we can define $\Psi_k:\mathscr{D}(\sigma_0,\beta_0)\to L(V_k,V_k)$ by

$$\Psi_k(\sigma,\beta):=\Psi(\sigma,\beta)|_{V_k}.$$

We note that V_k, $k\geqslant 1, k\in\mathbb{N}$, can be endowed with the natural complex structure $J:V_k\to V_k$ defined by (2.1). By extending the linearity of J to the vector space spanned over the field of complex numbers by $e^{ik\cdot}\cdot\epsilon_j:\mathbb{R}\ni t\to e^{ikt}\cdot\epsilon_j\in\mathbb{C}^{N+1}$, $j=1,2,\ldots,N+1$, we know that

$$\big\{e^{ik\cdot}\cdot\epsilon_j,\, J\big(e^{ik\cdot}\cdot\epsilon_j\big)\big\}_{j=1}^{N+1}=\big\{e^{ik\cdot}\cdot\epsilon_j, ie^{ik\cdot}\cdot\epsilon_j\big\}_{j=1}^{N+1}$$

is a basis of V_k, where $\{\epsilon_1,\epsilon_2,\dots,\epsilon_{N+1}\}$ denotes the standard basis of $\mathbb{R}^{N+1}$. Then we identify V_k with the vector space over the complex numbers spanned by $e^{ik\cdot}\cdot\epsilon_j$, $j=1,2,\dots,N+1$.

Then we have for $v_k\in V_k$, $k\in\mathbb{Z}$, $k\geqslant 1$,

$$\begin{aligned}\Psi_k(\sigma,\beta)v_k&=v_k-(L_0+K)^{-1}\bigg[\frac{1}{\beta}D_u\tilde N_0\big(u(\sigma),\sigma,\beta\big)+K\bigg]v_k\\&=v_k-\frac{1}{\beta}(L_0+K)^{-1}\left\{\begin{bmatrix}\partial_1 f(\sigma)&0\\ \partial_1 g(\sigma)&\partial_2 g(\sigma)\end{bmatrix}v_k+\begin{bmatrix}\partial_2 f(\sigma)&0\\0&0\end{bmatrix}(v_k)_{\beta z_\sigma}\right\},\end{aligned}$$

where $(v_k)_{\beta z_\sigma}=v_k(\cdot-\beta z_\sigma)$. By similar calculation for (3.27), we have, for $e^{ik\cdot}\epsilon_j\in V_k$,

$$\begin{aligned}\Psi_k(\sigma,\beta)\big(e^{ik\cdot}\epsilon_j\big)&=\frac{1}{ik\beta}\begin{bmatrix}ik\beta\mathrm{Id}-\partial_1 f(\sigma)-\partial_2 f(\sigma)e^{-ik\beta z_\sigma}&0\\-\partial_1 g(\sigma)&ik\beta-\partial_2 g(\sigma)\end{bmatrix}\cdot\big(e^{ik\cdot}\epsilon_j\big)\\&=\frac{1}{ik\beta}\Delta_{(u(\sigma),\sigma)}(ik\beta)\cdot\big(e^{ik\cdot}\epsilon_j\big),\end{aligned}$$

where the last equality follows from (3.3). Therefore, the matrix representation $[\Psi_k]$ of $\Psi_k(\sigma,\beta)$ with respect to the ordered $\mathbb{C}$-basis $\{e^{ik\cdot}\epsilon_j\}_{j=1}^{N+1}$ is given by

$$\frac{1}{ik\beta}\Delta_{(u(\sigma),\sigma)}(ik\beta).$$

For the application of Theorem 2.4, we now show that there exists some $k\in\mathbb{Z}$, $k\geqslant 1$, such that $\mu_k(u(\sigma_0),\sigma_0,\beta_0):=\deg_B(\det_{\mathbb{C}}[\Psi_k])\neq 0$.

Define $\Psi_H:\mathscr{D}(\sigma_0,\beta_0)\to\mathbb{R}^2\simeq\mathbb{C}$ by

$$\Psi_H(\sigma,\beta)=\det\Delta_{(u(\sigma),\sigma)}(i\beta).$$

The number $\mu_1(u(\sigma_0),\sigma_0,\beta_0)$ we defined in Lemma 2.3 can be written as follows:

$$\mu_1\big(u(\sigma_0),\sigma_0,\beta_0\big)=\epsilon\cdot\deg\big(\Psi_H,\mathscr{D}(\sigma_0,\beta_0)\big),$$

where $\epsilon=\operatorname{sign}\det\Psi_0(\sigma,\beta)$ for $(\sigma,\beta)\in\mathscr{D}(\sigma_0,\beta_0)$. For a constant map $v_0\in V_0$,

$$\Psi_0(\sigma,\beta)v_0=-\frac{1}{\beta}\begin{bmatrix}\partial_1 f(\sigma)+\partial_2 f(\sigma)&0\\ \partial_1 g(\sigma)&\partial_2 g(\sigma)\end{bmatrix}v_0.$$

Then, by (S3), we have $\epsilon\neq 0$ and therefore (A6) is satisfied.

Recall that α_0,β_0,δ and ε are chosen before (3.5). Define the function $H:[\sigma_0-\delta,\sigma_0+\delta]\times\overline{\Omega}\to\mathbb{R}^2\simeq\mathbb{C}$ by

$$H(\sigma,\alpha,\beta):=\det\Delta_{(u(\sigma),\sigma)}(\alpha+i\beta),$$

where $\Omega=(0,\alpha_0)\times(\beta_0-\varepsilon,\beta_0+\varepsilon)$, $\alpha_0=\alpha_0(\sigma_0,\beta_0)>0$. By the same argument for (3.4) and (3.5), we know that H satisfies all the conditions of Lemma 2.1 by the choice of $\alpha_0,\beta_0,\varepsilon$ and δ. So we have

$$\deg\big(\Psi_H,\mathscr{D}(\sigma_0,\beta_0)\big)=\gamma\big(u(\sigma_0),\sigma_0,\beta_0\big)\neq 0.$$

Thus, $\mu_1(u(\sigma_0), \sigma_0, \beta_0) \neq 0$ which, by Theorem 2.4, implies that $(u(\sigma_0), \sigma_0, \beta_0)$ is a bifurcation point of the system (3.62). Consequently, there exists a sequence of nonconstant periodic solutions $(u_n, \sigma_n, \beta_n) = (x_n, \tau_n, \sigma_n, \beta_n)$ such that $\sigma_n \to \sigma_0$, $\beta_n \to \beta_0$ as $n \to \infty$, and $(x_n(t), \tau_n(t))$ is a $2\pi/\beta_n$-periodic solution of (1.1) such that $\lim_{n\to+\infty} \|(x_n, \tau_n) - (x_{\sigma_0}, \tau_{\sigma_0})\| = 0$. □

4. Global bifurcation of FDEs with state-dependent delays

To use Theorem 2.5 to describe the maximal continuation of bifurcated periodic solutions with large amplitudes when the bifurcation parameter σ is far away from the bifurcation value, we need to prove that there is a lower bound for the periods of periodic solutions of system (1.1).

Lemma 4.1. *(See Vidossich [42].) Let X be a Banach space, $v : \mathbb{R} \to X$ be a p-periodic function with the following properties:*

(i) $v \in L^1_{\mathrm{loc}}(\mathbb{R}, X)$;
(ii) *there exists $U \in L^1([0, \frac{\mathrm{p}}{2}]; \mathbb{R}_+)$ such that $|v(t) - v(s)| \leqslant U(t-s)$ for almost every (in the sense of the Lebesgue measure) $s, t \in \mathbb{R}$ such that $s \leqslant t$, $t - s \leqslant \frac{\mathrm{p}}{2}$;*
(iii) $\int_0^{\mathrm{p}} v(t)\,dt = 0$.

Then

$$\mathrm{p}\|v\|_{L^\infty} \leqslant 2\int_0^{\frac{\mathrm{p}}{2}} U(t)\,dt.$$

We make the following assumption on system (1.1):

(S4) There exist constants $L_f > 0$, $L_g > 0$ such that

$$\begin{aligned}
\bigl|f(\theta_1, \theta_2, \sigma) - f(\bar\theta_1, \bar\theta_2, \sigma)\bigr| &\leqslant L_f\bigl(|\theta_1 - \bar\theta_1| + |\theta_2 - \bar\theta_2|\bigr),\\
\bigl|g(\gamma_1, \gamma_2, \sigma) - g(\overline{\gamma}_1, \overline{\gamma}_2, \sigma)\bigr| &\leqslant L_g\bigl(|\gamma_1 - \overline{\gamma}_1| + |\gamma_2 - \overline{\gamma}_2|\bigr)
\end{aligned}$$

for any $\theta_1, \theta_2, \bar\theta_1, \bar\theta_2, \gamma_1, \overline{\gamma}_1 \in \mathbb{R}^N$, $\gamma_2, \overline{\gamma}_2 \in \mathbb{R}$, $\sigma \in \mathbb{R}$.

Lemma 4.2. *Assume that system* (1.1) *satisfies the assumption* (S4). *If $u = (x, \tau)$ is a nonconstant periodic solution of* (1.1), *then the minimal period of u satisfies*

$$\mathrm{p} \geqslant \frac{4(|\dot x|_{L^\infty} + |\dot\tau|_{L^\infty})}{(2L_f + L_g)|\dot x|_{L^\infty} + L_g|\dot\tau|_{L^\infty} + L_f|\dot x|_{L^\infty}|\dot\tau|_{L^\infty}}.$$

Moreover, suppose $g(x, \tau, \sigma)$ satisfies that

(S5) *for every $\sigma \in \mathbb{R}$, there exists $L_0 > 0$ so that $-L_0 \leqslant g(x, \tau, \sigma) < 1$ for all $(x, \tau) \in \mathbb{R}^{N+1}$.*

Then the minimal period p of u satisfies

$$\mathrm{p} \geqslant \frac{4}{\max\{L_0, 1\} + 2(L_f + L_g)}.$$

Proof. Let $v(t) = \dot{u}(t)$. Then $\int_0^{\mathsf{p}} v(t)\,dt = 0$ since $u(t)$ is a p-periodic solution. For $s \leqslant t$, by (S4) and the Integral Mean Value Theorem, we have

$$\begin{aligned}
\bigl|v(t) - v(s)\bigr| &\leqslant \bigl|\dot{x}(t) - \dot{x}(s)\bigr| + \bigl|\dot{\tau}(t) - \dot{\tau}(s)\bigr| \\
&\leqslant L_f\bigl(\bigl|x(t) - x(s)\bigr| + \bigl|x\bigl(t - \tau(t)\bigr) - x\bigl(s - \tau(s)\bigr)\bigr|\bigr) \\
&\quad + L_g\bigl(\bigl|x(t) - x(s)\bigr| + \bigl|\tau(t) - \tau(s)\bigr|\bigr) \\
&\leqslant L_f|\dot{x}|_{L^\infty}(t - s) + L_f|\dot{x}|_{L^\infty}\bigl(t - s + \bigl|\tau(t) - \tau(s)\bigr|\bigr) \\
&\quad + L_g|\dot{x}|_{L^\infty}(t - s) + L_g|\dot{\tau}|_{L^\infty}(t - s) \\
&\leqslant \bigl[(2L_f + L_g)|\dot{x}|_{L^\infty} + L_g|\dot{\tau}|_{L^\infty} + L_f|\dot{x}|_{L^\infty} \cdot |\dot{\tau}|_{L^\infty}\bigr](t - s).
\end{aligned}$$

Let

$$U(t) = \bigl[(2L_f + L_g)|\dot{x}|_{L^\infty} + L_g|\dot{\tau}|_{L^\infty} + |\dot{x}|_{L^\infty} \cdot |\dot{\tau}|_{L^\infty}\bigr]t.$$

Then, by Lemma 4.1, we obtain

$$\mathsf{p}\bigl|(\dot{x}, \dot{\tau})\bigr|_{L^\infty} \leqslant 2\int_0^{\frac{\mathsf{p}}{2}} U(t)\,dt = \frac{\mathsf{p}^2}{4}\bigl[(2L_f + L_g)|\dot{x}|_{L^\infty} + L_g|\dot{\tau}|_{L^\infty} + |\dot{x}|_{L^\infty} \cdot |\dot{\tau}|_{L^\infty}\bigr].$$

Therefore,

$$\mathsf{p} \geqslant \frac{4|(\dot{x}, \dot{\tau})|_{L^\infty}}{(2L_f + L_g)|\dot{x}|_{L^\infty} + L_g|\dot{\tau}|_{L^\infty} + L_f|\dot{x}|_{L^\infty}|\dot{\tau}|_{L^\infty}}.$$

Moreover, if $-L_0 \leqslant g(x(t), \tau(t), \sigma) < 1$, then

$$|\dot{x}|_{L^\infty} \cdot |\dot{\tau}|_{L^\infty} \leqslant \max\{L_0, 1\}|\dot{x}|_{L^\infty},$$

and hence

$$\begin{aligned}
\mathsf{p} &\geqslant \frac{4|(\dot{x}, \dot{\tau})|_{L^\infty}}{(2L_f + L_g)|\dot{x}|_{L^\infty} + L_g|\dot{\tau}|_{L^\infty} + \max\{L_0, 1\}|\dot{x}|_{L^\infty}} \\
&\geqslant \frac{4|(\dot{x}, \dot{\tau})|_{L^\infty}}{(2L_f + L_g)|(\dot{x}, \dot{\tau})|_{L^\infty} + L_g|(\dot{x}, \dot{\tau})|_{L^\infty} + \max\{L_0, 1\}|(\dot{x}, \dot{\tau})|_{L^\infty}} \\
&= \frac{4}{\max\{L_0, 1\} + 2(L_f + L_g)}. \qquad \Box
\end{aligned}$$

The following result was first established by Mallet-Paret and Yorke [33] for ordinary differential equations and was extended to neutral equations by Wu [45].

Lemma 4.3. *Suppose that system* (1.1) *satisfies* (S1)–(S2) *and* (S4)–(S5). *Assume further that there exists a sequence of real numbers* $\{\sigma_k\}_{k=1}^{\infty}$ *such that*:

(i) *For each* k, *system* (1.1) *with* $\sigma = \sigma_k$ *has a nonconstant periodic solution* $u_k = (x_k, \tau_k) \in C(\mathbb{R}; \mathbb{R}^{N+1})$ *with the minimal period* $T_k > 0$;

(ii) $\lim_{k\to\infty}\sigma_k=\sigma_0\in\mathbb{R}$, $\lim_{k\to\infty}T_k=T_0<\infty$, *and* $\lim_{k\to\infty}\|u_k-u_0\|=0$, *where* $u_0:\mathbb{R}\to\mathbb{R}^{N+1}$ *is a constant map with the value* (x_0,τ_0).

Then (u_0,σ_0) *is a stationary state of* (1.1) *and there exists* $m\geqslant 1$, $m\in\mathbb{N}$ *such that* $\pm im2\pi/T_0$ *are the roots of the characteristic equation* (3.2) *with* $\sigma=\sigma_0$.

Proof. By Lemma 4.2 we conclude that $T_k\geqslant\frac{4}{\max\{L_0,1\}+2(L_f+L_g)}$ and therefore

$$T_0\geqslant\frac{4}{\max\{L_0,1\}+2(L_f+L_g)}>0.$$

Now we show that (u_0,σ_0) is a stationary state of (1.1). Since by (ii) $\lim_{k\to\infty}\sigma_k=\sigma_0$ and $\lim_{k\to\infty}\|u_k-u_0\|=0$, we only need show that the derivatives $\{\dot{u}_k\}_{k=1}^{+\infty}$ converge uniformly to the right-hand side of system (1.1). That is,

$$\begin{aligned}
&\left\|\begin{pmatrix} f(x_k,x_k(\cdot-\tau_k),\sigma_k)\\ g(x_k,\tau_k,\sigma_k)\end{pmatrix}-\begin{pmatrix} f(x_0,x_0,\sigma_0)\\ g(x_0,\tau_0,\sigma_0)\end{pmatrix}\right\|\\
&\quad\leqslant\big\|f\big(x_k,x_k(\cdot-\tau_k),\sigma_k\big)-f(x_0,x_0,\sigma_0)\big\|+\big\|g(x_k,\tau_k,\sigma_k)-g(x_0,\tau_0,\sigma_0)\big\|\\
&\quad\to 0,\quad \text{as } k\to+\infty.
\end{aligned}\tag{4.1}$$

Note that we have used $f(x_k,x_k(\cdot-\tau_k),\sigma_k)$ to denote the function $f(x_k(\cdot),x_k(\cdot-\tau_k),\sigma_k)$.

By (S1) and assumption (ii), we have $\lim_{k\to\infty}\|g(x_k,\tau_k,\sigma_k)-g(x_0,\tau_0,\sigma_0)\|=0$. By the Integral Mean Value Theorem, we have

$$\begin{aligned}
&\big\|f\big(x_k,x_k(\cdot-\tau_k),\sigma_k\big)-f(x_0,x_0,\sigma_0)\\
&\quad\leqslant\big\|f\big(x_k,x_k(\cdot-\tau_k),\sigma_k\big)-f\big(x_0,x_k(\cdot-\tau_k),\sigma_k\big)\big\|\\
&\qquad+\big\|f\big(x_0,x_k(\cdot-\tau_k),\sigma_k\big)-f(x_0,x_0,\sigma_k)\big\|+\big|f(x_0,x_0,\sigma_k)-f(x_0,x_0,\sigma_0)\big|\\
&\quad\leqslant\big\|\partial_1 f\big(x_k+s(x_k-x_0),x_k(\cdot-\tau_k),\sigma_k\big)\big\|\cdot\|x_k-x_0\|\\
&\qquad+\big\|\partial_2 f\big(x_0,x_0+s\big(x_k(\cdot-\tau_k)-x_0\big),\sigma_k\big)\big\|\cdot\|x_k-x_0\|\\
&\qquad+\big|f(x_0,x_0,\sigma_k)-f(x_0,x_0,\sigma_0)\big|
\end{aligned}\tag{4.2}$$

for some $s\in[0,1]$. By (S1), assumption (ii) and by an argument similar to that of (3.16) we know that there exists $\tilde{L}_4>0$ such that

$$\begin{cases}\big\|\partial_1 f\big(x_k+s(x_k-x_0),x_k(\cdot-\tau_k),\sigma_k\big)\big\|\leqslant\tilde{L}_4,\\ \big\|\partial_2 f\big(x_0,x_0+s\big(x_k(\cdot-\tau_k)-x_0\big),\sigma_k\big)\big\|\leqslant\tilde{L}_4.\end{cases}\tag{4.3}$$

Then by (4.2), (4.3) and assumption (ii), we have

$$\begin{aligned}
&\lim_{k\to\infty}\big\|f\big(x_k,x_k(\cdot-\tau_k),\sigma_k\big)-f(x_0,x_0,\sigma_0)\big\|\\
&\quad\leqslant\lim_{k\to\infty}\big(2\tilde{L}_4\|x_k-x_0\|+\big|f(x_0,x_0,\sigma_k)-f(x_0,x_0,\sigma_0)\big|\big)\\
&\quad=0.
\end{aligned}$$

This completes the proof of (4.1). Therefore, $(u_0, \sigma_0) = (x_{\sigma_0}, \tau_{\sigma_0}, \sigma_0)$ is the stationary state of (1.1) with $\sigma = \sigma_0$.

Next, we show that the following linear system

$$\dot{v}(t) = \begin{bmatrix} \partial_1 f(\sigma_0) & 0 \\ \partial_1 g(\sigma_0) & \partial_2 g(\sigma_0) \end{bmatrix} v(t) + \begin{bmatrix} \partial_2 f(\sigma_0) & 0 \\ 0 & 0 \end{bmatrix} v(t-\tau_0) \tag{4.4}$$

has a nonconstant periodic solution.

For $\rho \in (0,1)$, define

$$\epsilon_{k,\rho} = \max_{t\in\mathbb{R}} \bigl|u_k(t+\rho T_k) - u_k(t)\bigr|, \tag{4.5}$$

$$v_k(t) = \epsilon_{k,\rho}^{-1}\bigl[u_k(t+\rho T_k) - u_k(t)\bigr]. \tag{4.6}$$

Then, $\|v_k\| = 1$ and $v_k(t) := (y_k(t), z_k(t))$ satisfies

$$\dot{v}_k(t) = \begin{bmatrix} \partial_1 f(\sigma_0) & 0 \\ \partial_1 g(\sigma_0) & \partial_2 g(\sigma_0) \end{bmatrix} v_k(t) + \begin{bmatrix} \partial_2 f(\sigma_0) & 0 \\ 0 & 0 \end{bmatrix} v_k(t-\tau_0) + \begin{pmatrix} \delta_{1k}(t) \\ \delta_{2k}(t) \end{pmatrix} \tag{4.7}$$

where

$$\begin{cases} \delta_{1k}(t) = \epsilon_{k,\rho}^{-1}\bigl[f\bigl(x_k(t+\rho T_k), x_k\bigl(t+\rho T_k - \tau_k(t+\rho T_k)\bigr), \sigma_k\bigr) \\ \qquad - f\bigl(x_k(t), x_k\bigl(t-\tau_k(t)\bigr), \sigma_k\bigr) - \partial_1 f(\sigma_0)\bigl(x_k(t+\rho T_k) - x_k(t)\bigr) \\ \qquad - \partial_2 f(\sigma_0)\bigl(x_k(t+\rho T_k - \tau_0) - x_k(t-\tau_0)\bigr)\bigr], \\ \delta_{2k}(t) = \epsilon_{k,\rho}^{-1}\bigl[g\bigl(x_k(t+\rho T_k), \tau_k(t+\rho T_k), \sigma_k\bigr) - g\bigl(x_k(t), \tau_k(t), \sigma_k\bigr) \\ \qquad - \partial_1 g(\sigma_0)\bigl(x_k(t+\rho T_k) - x_k(t)\bigr) - \partial_2 g(\sigma_0)\bigl(\tau_k(t+\rho T_k) - \tau_k(t)\bigr)\bigr]. \end{cases}$$

We now show that $|\delta_{1k}(t)| \to 0$, $|\delta_{2k}(t)| \to 0$ as $k \to +\infty$ uniformly for $t \in \mathbb{R}$. Indeed, by (4.6) and the Integral Mean Value Theorem, we have

$$\begin{aligned} \bigl|\delta_{1k}(t)\bigr| &= \epsilon_{k,\rho}^{-1}\bigl|f\bigl(x_k(t+\rho T_k), x_k\bigl(t+\rho T_k - \tau_k(t+\rho T_k)\bigr), \sigma_k\bigr) \\ &\quad - f\bigl(x_k(t), x_k\bigl(t-\tau_k(t)\bigr), \sigma_k\bigr) - \partial_1 f(\sigma_0)\bigl(x_k(t+\rho T_k) - x_k(t)\bigr) \\ &\quad - \partial_2 f(\sigma_0)\bigl(x_k\bigl(t+\rho T_k - \tau_k(t+\rho T_k)\bigr) - x_k\bigl(t-\tau_k(t)\bigr)\bigr) \\ &\quad - \partial_2 f(\sigma_0)\bigl(x_k(t+\rho T_k - \tau_0) - x_k(t-\tau_0)\bigr) \\ &\quad + \partial_2 f(\sigma_0)\bigl(x_k\bigl(t+\rho T_k - \tau_k(t+\rho T_k)\bigr) - x_k\bigl(t-\tau_k(t)\bigr)\bigr)\bigr| \\ &= \Biggl|\int_0^1 \bigl(\partial_1 f_k(\sigma_k,\theta,\rho)(t) - \partial_1 f(\sigma_0)\bigr)\, d\theta\, y_k(t) \\ &\quad + \int_0^1 \bigl(\partial_2 f_k(\sigma_k,\theta,\rho)(t) - \partial_1 f(\sigma_0)\bigr)\, d\theta\, y_k\bigl(t-\tau_k(t)\bigr) \end{aligned}$$

$$
\begin{aligned}
&+ \partial_2 f(\sigma_0)\big(x_k\big(t+\rho T_k - \tau_k(t+\rho T_k)\big) - x_k(t+\rho T_k - \tau_0) \\
&- \big(x_k\big(t-\tau_k(t)\big) - x_k(t-\tau_0)\big)\big)\Big| \\
:= {}& |A_1 + A_2 + A_3|,
\end{aligned}
$$

where

$$
\partial_1 f_k(\sigma_k, \theta, \rho)(t) = \partial_1 f\big(x_k(t) + \theta\big(x_k(t+\rho T_k) - x_k(t)\big), x_k\big(t+\rho T_k - \tau_k(t+\rho T_k)\big), \sigma_k\big),
$$

$$
\partial_2 f_k(\sigma_k, \theta, \rho)(t) = \partial_2 f\big(x_k(t), x_k\big(t-\tau_k(t)\big) + \theta\big(x_k\big(t+\rho T_k - \tau_k(t+\rho T_k)\big) - x_k\big(t-\tau_k(t)\big)\big), \sigma_k\big),
$$

$$
A_1 = \int_0^1 \big(\partial_1 f_k(\sigma_k, \theta, \rho)(t) - \partial_1 f(\sigma_0)\big)\, d\theta\, y_k(t),
$$

$$
A_2 = \int_0^1 \big(\partial_2 f_k(\sigma_k, \theta, \rho)(t) - \partial_2 f(\sigma_0)\big)\, d\theta\, y_k\big(t-\tau_k(t)\big),
$$

$$
A_3 = \partial_2 f(\sigma_0)\big(x_k\big(t+\rho T_k - \tau_k(t+\rho T_k)\big) - x_k(t+\rho T_k - \tau_0) - \big(x_k\big(t-\tau_k(t)\big) - x_k(t-\tau_0)\big)\big).
$$

Recall that $\partial_i f(\sigma_0)$ and $\partial_i g(\sigma_0)$, $i = 1, 2$, are defined in (3.1). By assumption (ii), we know that $(x_k(t) + \theta(x_k(t+\rho T_k) - x_k(t)), x_k(t+\rho T_k - \tau_k(t+\rho T_k)), \sigma_k)$ converges to $(x_{\sigma_0}, x_{\sigma_0}, \sigma_0)$ in $C(\mathbb{R}; \mathbb{R}^N) \times C(\mathbb{R}; \mathbb{R}^N) \times \mathbb{R}$ uniformly for all $\theta \in [0, 1]$, $\rho \in (0, 1)$. By (S1) we know that the map $f : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \ni (\theta_1, \theta_2, \sigma) \to f(\theta_1, \theta_2, \sigma) \in \mathbb{R}^N$ is C^2 in (θ_1, θ_2), and $\partial_1 f : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \ni (\theta_1, \theta_2, \sigma) \to \partial_1 f(\theta_1, \theta_2, \sigma) \in L(\mathbb{R}^N, \mathbb{R}^N)$ is C^1 in σ. Then there exists a constant $\tilde{L}_5 > 0$ so that

$$
\begin{aligned}
&\big|\partial_1 f_k(\sigma_k, \theta, \rho)(t) - \partial_1 f(\sigma_0)\big| \\
&\quad \leqslant \tilde{L}_5 \big|\big(x_k(t) + \theta\big(x_k(t+\rho T_k) - x_k(t)\big), x_k\big(t+\rho T_k - \tau_k(t+\rho T_k)\big), \sigma_k\big) - (x_{\sigma_0}, \tau_{\sigma_0}, \sigma_0)\big|
\end{aligned}
$$

for all $t \in \mathbb{R}$, $k \in \mathbb{N}$, $\theta \in [0, 1]$ and $\rho \in (0, 1)$. Therefore, we have

$$
\lim_{k\to+\infty} \big\|\partial_1 f_k(\sigma_k, \theta, \rho) - \partial_1 f(\sigma_0)\big\| = 0, \tag{4.8}
$$

uniformly for $\theta \in [0, 1]$, $\rho \in (0, 1)$. Then by (4.6), (4.8) and the fact that $\|y_k\| = 1$, we have

$$
\|A_1\| \to 0 \quad \text{as } k \to +\infty. \tag{4.9}
$$

Similarly we have

$$
\|A_2\| \to 0 \quad \text{as } k \to +\infty. \tag{4.10}
$$

Also, by the Integral Mean Value Theorem, we have

$$
|A_3| \leqslant \big\|\partial_2 f(\sigma_0)\big\| \|\dot{x}_k\| \cdot \big|\tau_k(t+\rho T_k) - \tau_0)\big| + \big\|\partial_2 f(\sigma_0)\big\| \|\dot{x}_k\| \cdot \big|\tau_k(t) - \tau_0\big|. \tag{4.11}
$$

By assumption (ii) and system (1.1) we know that $\|\dot{x}_k\|$ is uniformly bounded for all $k \in \mathbb{N}$ and $\|\tau_k - \tau_0\| \to 0$ as $k \to +\infty$. Then it follows from (4.11) that

$$\|A_3\| \to 0 \quad \text{as } k \to +\infty. \tag{4.12}$$

Therefore, we have from (4.9), (4.10) and (4.12) that

$$\|\delta_{1k}\| \to 0 \quad \text{as } k \to +\infty. \tag{4.13}$$

Similarly, we have

$$\|\delta_{2k}\| \to 0 \quad \text{as } k \to +\infty. \tag{4.14}$$

By (4.7), (4.13), (4.14) and the fact that $\|v_k\| = 1$, we know that there exists $\tilde{L}_6 > 0$ such that $\|\dot{v}_k\| \leqslant \tilde{L}_6$ for all $k \in \mathbb{N}$. Also, by assumption (ii), the set of periods $\{T_k\}_{k=1}^{+\infty}$ is bounded. Then by the Arzela–Ascoli Theorem, $\{v_k\}_{k=1}^{+\infty}$ has a convergent subsequence, denoted by $\{v_{k_j}\}_{j=1}^{+\infty}$. Let

$$v_\rho(t) = \lim_{j \to +\infty} v_{k_j}(t). \tag{4.15}$$

Then v_ρ is a periodic solution of (4.4) with period T_0. Since $\|v_k\| = 1$ and the average value of each v_k is zero, the same is true for v_ρ. So v_ρ is a nonconstant T_0-periodic solution of (4.4). Then by Lemma 3.3, there exists $m \geqslant 1$, $m \in \mathbb{N}$, such that $\pm im2\pi/T_0$ are characteristic values of (3.2). This completes the proof. $\square$

Now we can describe the relation between $2\pi/\beta_k$ and the minimal period of u_k in Theorem 3.6.

Theorem 4.4. *Assume* (S1)–(S5) *hold. In Theorem* 3.6, *every limit point of the minimal period of* $u_k = (x_k, \tau_k)$ *as* $k \to +\infty$ *is contained in the set*

$$\left\{ \frac{2\pi}{(n\beta_0)} \colon \pm imn\beta_0 \text{ are characteristic values of } (u_0, \sigma_0),\ m, n \geqslant 1,\ m, n \in \mathbb{N} \right\}.$$

Moreover, if $\pm imn\beta_0$ *are not characteristic values of* (u_0, σ_0) *for any integers* $m, n \in \mathbb{N}$ *such that* $mn > 1$, *then* $2\pi/\beta_k$ *is the minimal period of* $u_k(t)$ *and* $2\pi/\beta_k \to 2\pi/\beta_0$ *as* $k \to \infty$.

Proof. Let T_k denote the minimal period of $u_k(t)$. Then there exists a positive integer n_k such that $2\pi/\beta_k = n_k T_k$. Since $T_k \leqslant 2\pi/\beta_k \to 2\pi/\beta_0$ as $k \to \infty$, there exists a subsequence $\{T_{k_j}\}_{j=1}^{\infty}$ and T_0 such that $T_0 = \lim_{j\to\infty} T_{k_j}$. Since $2\pi/\beta_{k_j} \to 2\pi/\beta_0$, $T_{k_j} \to T_0$ as $j \to \infty$, n_{k_j} is identical to a constant n for k large enough. Therefore, $2\pi/\beta_0 = nT_0$. Thus $T_{k_j} \to 2\pi/(n\beta_0)$ as $j \to \infty$. By Lemma 4.3, $\pm im2\pi/T_0 = \pm imn\beta_0$ are characteristic values of (u_0, σ_0) for some $m \geqslant 1$, $m \in \mathbb{N}$.

Moreover, if $\pm imn\beta_0$ are not characteristic values of (u_0, σ_0) for any integers $m \in \mathbb{N}$ and $n \in \mathbb{N}$ with $mn > 1$, then $m = n = 1$. Therefore, for k large enough $n_{k_j} = 1$ and $2\pi/\beta_k = T_k$ is the minimal period of $u_k(t)$ and $2\pi/\beta_k \to 2\pi/\beta_0$ as $k \to \infty$. This completes the proof. $\square$

The following lemma shows that we can locate all the possible Hopf bifurcation points of system (1.1) with state-dependent delay at the centers of its corresponding formal linearization.

Lemma 4.5. *Assume* (S1)–(S3) *hold. If* (u_0, σ_0) *is a Hopf bifurcation point of system* (1.1), *then it is a center of* (3.1).

Proof. If (u_0, σ_0) is a Hopf bifurcation point of system (1.1), then there exist a sequence

$$\big\{(u_k, \sigma_k, T_k)\big\}_{k=1}^{+\infty} \subseteq C\big(\mathbb{R}; \mathbb{R}^{N+1}\big) \times \mathbb{R}^2$$

and $T_0 \geqslant 0$ such that $\lim_{k\to+\infty} \|(u_k, \sigma_k, T_k) - (u_0, \sigma_0, T_0)\| = 0$, where (u_k, σ_k) is a nonconstant T_k-periodic solution of system (1.1). Our strategy here is to show that the system

$$\dot{v}(t) = \begin{bmatrix} \partial_1 f(\sigma_0) & 0 \\ \partial_1 g(\sigma_0) & \partial_2 g(\sigma_0) \end{bmatrix} v(t) + \begin{bmatrix} \partial_2 f(\sigma_0) & 0 \\ 0 & 0 \end{bmatrix} v(t - \tau_{\sigma_0}) \tag{4.16}$$

has a nonconstant periodic solution, and hence $u_0 = (x_{\sigma_0}, \tau_{\sigma_0})$ is a center of (3.1).

By (S1) and the Integral Mean Value Theorem we have

$$\begin{aligned} \dot{u}_k(t) = {} & \int_0^1 \begin{bmatrix} \partial_1 f_k(\sigma_k, s)(t) & 0 \\ \partial_1 g_k(\sigma_k, s)(t) & \partial_2 g_k(\sigma_k, s)(t) \end{bmatrix} ds \begin{pmatrix} x_k(t) - x_{\sigma_k} \\ \tau_k(t) - \tau_{\sigma_k} \end{pmatrix} \\ & + \int_0^1 \begin{bmatrix} \partial_2 f_k(\sigma_k, s)(t) & 0 \\ 0 & 0 \end{bmatrix} ds \begin{pmatrix} x_k(t - \tau_k(t)) - x_{\sigma_k} \\ \tau_k(t - \tau_k(t)) - \tau_{\sigma_k} \end{pmatrix}, \end{aligned} \tag{4.17}$$

where

$$\begin{aligned} \partial_1 f_k(\sigma_k, s)(t) &:= \partial_1 f\big(x_{\sigma_k} + s\big(x_k(t) - x_{\sigma_k}\big), x_{\sigma_k} + s\big(x_k\big(t - \tau_k(t)\big) - x_{\sigma_k}, \sigma_k\big)\big), \\ \partial_2 f_k(\sigma_k, s)(t) &:= \partial_2 f\big(x_{\sigma_k} + s\big(x_k(t) - x_{\sigma_k}\big), x_{\sigma_k} + s\big(x_k\big(t - \tau_k(t)\big) - x_{\sigma_k}, \sigma_k\big)\big), \\ \partial_1 g_k(\sigma_k, s)(t) &:= \partial_1 g\big(x_{\sigma_k} + s\big(x_k(t) - x_{\sigma_k}\big), \tau_{\sigma_k} + s\big(\tau_k(t) - \tau_{\sigma_k}\big), \sigma_k\big), \\ \partial_2 g_k(\sigma_k, s)(t) &:= \partial_2 g\big(x_{\sigma_k} + s\big(x_k(t) - x_{\sigma_k}\big), \tau_{\sigma_k} + s\big(\tau_k(t) - \tau_{\sigma_k}\big), \sigma_k\big). \end{aligned}$$

Put

$$v_k(t) = \frac{u_k(t) - \eta(\sigma_k)}{\|u_k - \eta(\sigma_k)\|}. \tag{4.18}$$

Then we have

$$v_k\big(t - \tau_k(t)\big) = \frac{u_k(t - \tau_k(t)) - \eta(\sigma_k)}{\|u_k - \eta(\sigma_k)\|}. \tag{4.19}$$

By (4.17) and (4.19) we have

$$\begin{aligned} \dot{v}_k(t) = {} & \int_0^1 \begin{bmatrix} \partial_1 f_k(\sigma_k, s)(t) & 0 \\ \partial_1 g_k(\sigma_k, s)(t) & \partial_2 g_k(\sigma_k, s)(t) \end{bmatrix} ds\, v_k(t) \\ & + \int_0^1 \begin{bmatrix} \partial_2 f_k(\sigma_k, s)(t) & 0 \\ 0 & 0 \end{bmatrix} ds\, v_k\big(t - \tau_k(t)\big). \end{aligned} \tag{4.20}$$

We claim that there exists a convergent subsequence of $\{v_k\}_{k=1}^{+\infty}$. Indeed, note that by (S2) $\dot{\tau}_k(t) = g(x_k(t), \tau_k(t), \sigma_k) < \frac{L}{L+1} < 1$ implies that $\lim_{t\to+\infty}[t - \tau_k(t)] = +\infty$. Then by (4.18) and (4.19), we have $\|v_k\| = 1$, $\|v_k(\cdot - \tau_k(\cdot))\| = 1$. Note that by (S1) and a similar argument to (3.38), we have

$$\begin{cases} \lim\limits_{k\to+\infty} \big\|\partial_1 f_k(\sigma_k, s) - \partial_1 f(\sigma_0)\big\| = 0, \\ \lim\limits_{k\to+\infty} \big\|\partial_2 f_k(\sigma_k, s) - \partial_1 f(\sigma_0)\big\| = 0, \\ \lim\limits_{k\to+\infty} \big\|\partial_1 g_k(\sigma_k, s) - \partial_1 g(\sigma_0)\big\| = 0, \\ \lim\limits_{k\to+\infty} \big\|\partial_2 g_k(\sigma_k, s) - \partial_1 g(\sigma_0)\big\| = 0, \end{cases} \tag{4.21}$$

uniformly for $s \in [0, 1]$. It is clear from (4.21) that $\|\partial_1 f_k(\sigma_k, s)\|$, $\|\partial_2 f_k(\sigma_k, s)\|$, $\|\partial_1 g_k(\sigma_k, s)\|$ and $\|\partial_2 g_k(\sigma_k, s)\|$ are all uniformly bounded for any $k \in \mathbb{N}$ and $s \in [0, 1]$. It follows from (4.20) and (4.21) that there exists $\tilde{L}_7 > 0$ such that $\|\dot{v}_k\| < \tilde{L}_7$ for any $k \in \mathbb{N}$. Also, we note that the period T_k of (u_k, σ_k) is uniformly bounded for all $k \in \mathbb{N}$. Then by the Arzela–Ascoli Theorem, there exists a convergent subsequence $\{v_{k_j}\}_{j=1}^{+\infty}$ of $\{v_k\}_{k=1}^{+\infty}$. That is, there exists $v^* \in \{v \in C(\mathbb{R}; \mathbb{R}^{N+1})\colon \|v\| = 1\}$ such that

$$\lim_{j\to+\infty} \big\|v_{k_j} - v^*\big\| = 0. \tag{4.22}$$

By the Integral Mean Value Theorem, we have

$$\begin{aligned} \big|v_{k_j}\big(t - \tau_{k_j}(t)\big) - v_{k_j}(t - \tau_{\sigma_0})\big| &\leqslant \|\dot{v}_{k_j}\| \cdot \big|\tau_{k_j}(t) - \tau_{\sigma_0}\big| \\ &\leqslant \tilde{L}_7 \big|\tau_{k_j}(t) - \tau_{\sigma_0}\big|. \end{aligned} \tag{4.23}$$

By assumption we have

$$\lim_{j\to+\infty} \|\tau_{k_j} - \tau_{\sigma_0}\| = 0. \tag{4.24}$$

Then by (4.23) and (4.24) we have

$$\lim_{j\to+\infty} \big\|v_{k_j}\big(\cdot - \tau_{k_j}(\cdot)\big) - v_{k_j}(\cdot - \tau_{\sigma_0})\big\| = 0. \tag{4.25}$$

From (4.22) and (4.25) we have

$$\lim_{j\to+\infty} \big\|v_{k_j}\big(\cdot - \tau_{k_j}(\cdot)\big) - v^*(\cdot - \tau_{\sigma_0})\big\| = 0. \tag{4.26}$$

It follows from (4.21), (4.22) and (4.26) that the right-hand side of (4.20) converges uniformly to the right-hand side of (4.16). Therefore, we obtain that v^* is continuously differentiable with

$$\lim_{j\to+\infty} \big|\dot{v}_j(t) - \dot{v}^*(t)\big| = 0$$

and

$$\dot{v}^*(t) = \begin{bmatrix} \partial_1 f(\sigma_0) & 0 \\ \partial_1 g(\sigma_0) & \partial_2 g(\sigma_0) \end{bmatrix} v^*(t) + \begin{bmatrix} \partial_2 f(\sigma_0) & 0 \\ 0 & 0 \end{bmatrix} v^*(t - \tau_{\sigma_0}). \tag{4.27}$$

By (S3), we know that the matrix

$$\begin{bmatrix} \partial_1 f(\sigma_0) + \partial_2 f(\sigma_0) & 0 \\ \partial_1 g(\sigma_0) & \partial_2 g(\sigma_0) \end{bmatrix}$$

is nonsingular. Taking integrals on both sides of (4.27) over one period of v^*, we obtain that $v^* \in \{v\colon \|v\| = 1\}$ has zero integral average value in one period. So v^* is a nonconstant periodic solution of (4.27). Therefore, $(v^* + u_0, \sigma_0)$ is a nonconstant periodic solution of (3.1).

Then, by Lemma 3.3, (u_0, σ_0) is a center of (3.1). $\square$

Now we are able to consider the global Hopf bifurcation problem of system (1.1). Letting $(x(t), \tau(t)) = (y(\frac{2\pi}{\mathsf{p}}t), z(\frac{2\pi}{\mathsf{p}}t))$, we can reformulate the problem as a problem of finding 2π-period solutions to the following equation:

$$\dot{u}(t) = \frac{\mathsf{p}}{2\pi} N_0\big(u(t), \sigma, 2\pi/\mathsf{p}\big), \tag{4.28}$$

where $u(t) = (y(t), z(t))$. Accordingly, the formal linearization (3.1) becomes

$$\dot{u}(t) = \frac{\mathsf{p}}{2\pi} \tilde{N}_0\big(u(t), \sigma, 2\pi/\mathsf{p}\big). \tag{4.29}$$

Using the same notations as in the proof of Theorem 3.6, we can define $\mathscr{N}_0(u, \sigma, \mathsf{p}) = N_0(u, \sigma, 2\pi/\mathsf{p})$, $\tilde{\mathscr{N}_0}(u, \sigma, \mathsf{p}) = \tilde{N}_0(u, \sigma, 2\pi/\mathsf{p})$.

Then the following system

$$L_0 u = \frac{\mathsf{p}}{2\pi} \mathscr{N}_0(u, \sigma, \mathsf{p}), \quad \mathsf{p} > 0, \tag{4.30}$$

is equivalent to (4.28) and

$$L_0 u = \frac{\mathsf{p}}{2\pi} \tilde{\mathscr{N}_0}(u, \sigma, \mathsf{p}), \quad \mathsf{p} > 0, \tag{4.31}$$

is equivalent to (4.29). Let $\mathcal{S}$ denote the closure of the set of all nontrivial periodic solutions of system (4.30) in the space $V \times \mathbb{R} \times \mathbb{R}_+$, where $\mathbb{R}_+$ is the set of all nonnegative reals. It follows from Lemma 4.2 that the constant solution $(u_0, \sigma_0, 0)$ does not belong to this set. Consequently, we can assume that problem (4.30) is well posed on the whole space $V \times \mathbb{R}^2$, in the sense that if $\mathcal{S}$ exists in $V \times \mathbb{R}^2$, then it must be contained in $V \times \mathbb{R} \times \mathbb{R}_+$.

On the other hand, assume (S3) holds at every center of (4.31). Then, from the proof of Theorem 3.6 we know that the assumptions (S1)–(S3) are sufficient for the systems (4.30) and (4.31) to satisfy the conditions (A1)–(A6). Also, under the same assumptions, Lemma 4.5 implies (A7) and Lemma 3.5 implies (A8). Then by Theorem 2.5, we obtain the following global Hopf bifurcation theorem for system (4.30) with state-dependent delay.

Theorem 4.6. *Suppose that system* (1.1) *satisfies* (S1)–(S5) *and* (S3) *holds at every center of* (4.31)*. Assume that all the centers of* (4.31) *are isolated. Let M be the set of trivial periodic solutions of* (4.30) *and M is complete. If $(u_0, \sigma_0, \mathsf{p}_0) \in M$ is a bifurcation point, then either the connected component $C(u_0, \sigma_0, \mathsf{p}_0)$ of $(u_0, \sigma_0, \mathsf{p}_0)$ in $\mathcal{S}$ is unbounded, or*

$$C(u_0, \sigma_0, \mathsf{p}_0) \cap M = \big\{(u_0, \sigma_0, \mathsf{p}_0), (u_1, \sigma_1, \mathsf{p}_1), \ldots, (u_q, \sigma_q, \mathsf{p}_q)\big\},$$

where $\mathsf{p}_i \in \mathbb{R}_+$, $(u_i, \sigma_i, \mathsf{p}_i) \in M$, $i = 0, 1, 2, \ldots, q$. *Moreover, in the latter case, we have*

$$\sum_{i=0}^{q} \epsilon_i \gamma(u_i, \sigma_i, 2\pi/\mathsf{p}_i) = 0,$$

where $\gamma(u_i, \sigma_i, 2\pi/\mathsf{p}_i)$ *is the crossing number of* $(u_i, \sigma_i, \mathsf{p}_i)$ *defined by* (3.5) *and*

$$\epsilon_i = \operatorname{sgn}\det\begin{bmatrix} \partial_1 f(\sigma_i) + \partial_2 f(\sigma_i) & 0 \\ \partial_1 g(\sigma_i) & \partial_2 g(\sigma_i) \end{bmatrix}.$$

5. Remarks

A local Hopf bifurcation theory for FDEs with state-dependent delays was developed by Eichmann [14], where the existence of a local Hopf bifurcation is guaranteed by a transversality condition. This transversality implies that the crossing number defined in our paper is not zero, and hence the existence of a local Hopf bifurcation is also established in Theorem 3.6. Note that even in the case of a constant delay, one can have nontrivial crossing number while the transversality condition is not satisfied. Note also that the work of Eichmann gives more information about the local Hopf bifurcation such as smoothness of the bifurcation curve with respect to the parameter.

Earlier results on the existence of periodic solutions for FDEs with state-dependent delay include the work by Smith [40] that considered bifurcations of periodic solutions from a stationary state for a system of integral equations with state-dependent delay, and the work on the existence of periodic solutions by Nussbaum, Mallet-Paret and Paraskevopoulos [32]. These studies address the aspect of global continuation of Hopf bifurcations of periodic solutions, specially the existence of periodic solutions where the bifurcation parameter is away from the critical value where a local Hopf bifurcation is born. The work of Nussbaum, Mallet-Paret and Paraskevopoulos [32] focuses on a prototype class of state-dependent delay differential equations with negative feedback, and provides some detailed information of the so-called slowly oscillating periodic solutions. See [3,7,23,26–31,34–36,39–41,43] for other relevant studies on periodic solutions of state-dependent delay differential equations. In comparison, our results here provide a general tool and framework to study the Hopf bifurcation problem and, in particular, the global continuation of local bifurcation of periodic solutions of differential equations with state-dependent delay from an equivariant degree point of view. Our global bifurcation theory is, in principle, applicable to general system of FDEs with state-dependent delay and even some neutral equations with appropriate further development (see [15] for some work on equivalent local Hopf bifurcations of neutral equations with constant delays). How this theory is applied to specific systems will be illustrated in future studies, and whether this theory can be extended to other types of state-dependent delay differential equations requires further examination.

Like the work of Nussbaum, Mallet-Paret and Paraskevopoulos [32], our approach is based on a homotopy argument for calculating a topological index—the S^1-degree. As such, much of the effort has been dedicated to justify that the detection of Hopf bifurcation can be achieved through the formal linearization technique: the state-dependent delay $\tau(t)$ in $x(t-\tau(t))$ is first fixed at a given stationary state, then the resulting nonlinear system with the frozen constant delay is linearized. This linearization technique is utilized in the functional analytic setting that converts the Hopf bifurcation problem of system (1.1) to solving an operator equation (2.2) involving S^1-equivariant maps with two parameters, in the space of periodic functions with a fixed period. Implicitly used is the C^1-smoothness of the operator defined in Lemma 3.4 in the space V (the space of periodic functions with the fixed period 2π). The formal linearization leads to this operator naturally in the space of continuously differentiable periodic functions with the period 2π, and the fact that this operator can be extended to a bounded operator in the space V is essential in our homotopy argument. This technique of extending the linearized operator of a state-dependent delay differential equations from C^1 space to C space has previously been used in other contexts, see, for example, Nussbaum, Mallet-Paret and Paraskevopoulos [32,22,44] and the survey paper [18]. See [9,16,17,20] for additional references on linearization stability principles.

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